Eur. Phys. J. C 24, 595–612 (2002)
Digital Object Identifier (DOI) 10.1007/s10052-002-0968-0

THE EUROPEAN PHYSICAL JOURNAL C

Discrete symmetry and GUT breaking

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Received: 12 February 2002 /
Published online: 14 June 2002 –

Abstract. We study the supersymmetric GUT models in which the supersymmetry and GUT gauge symmetry can be broken by a discrete symmetry. First, with the ansatz that there exist discrete symmetries in the branes' neighborhoods, we discuss the general reflection Z_2 symmetries and GUT breaking on $M^4 \times M^1$ and $M^4 \times M^1 \times M^1$. In those models, the extra dimensions can be large and the KK states can be set arbitrarily heavy. Second, considering that the extra space manifold is the annulus A^2 or the disc D^2, we can define any Z_n symmetry and break any 6-dimensional $N = 2$ supersymmetric $SU(M)$ models down to the 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{M-4}$ models for the zero modes. In particular, there might exist the interesting scenario on $M^4 \times A^2$ where just a few KK states are light, while the others are relatively heavy. Third, we discuss the complete global discrete symmetries on $M^4 \times T^2$ and study the GUT breaking.

1 Introduction

Grand unified theory (GUT) gives us a simple and elegant understanding of the quantum numbers of the quarks and leptons, and the success of gauge coupling unification in the minimal supersymmetric standard model strongly supports this idea. The grand unified theory at a high energy scale has been widely accepted now, but there are some problems in GUT: the grand unified gauge symmetry breaking mechanism, the doublet–triplet splitting problem, the proton decay, etc.

As we know, one obvious approach to break GUT gauge symmetry is the Higgs mechanism [1], which is discussed extensively in phenomenology. Another approach is the one of spin connection embedding, which is used in the weakly coupled heterotic string $E_8 \times E_8$, and M-theory on S^1/Z_2 [2]. Because the Calabi–Yau manifold has $SU(3)$ holonomy, the observable E_8 gauge group can be broken down to E_6 by spin connection embedding. In addition, the GUT gauge symmetry can be broken down to a low energy subgroup by means of Wilson lines, provided that the fundamental group of the extra space manifold or orbifold is non-trivial [3].

Recently, a new scenario to explain the above questions in GUT has been suggested by Kawamura [4–6], and further discussed in a lot of papers [7,8]. The key point is that the GUT gauge symmetry exists in 5 or higher dimensions and is broken down to the 4-dimensional $N = 1$ supersymmetric standard model like gauge symmetry for the zero modes due to the discrete symmetries in the branes' neighborhoods, which become the non-trivial orbifold projections on the multiplets and gauge generators in GUT. Therefore, we would like to call this the discrete symmetry approach. Attractive models have been constructed explicitly where the supersymmetric 5-dimensional and 6-dimensional GUT models are broken down to the 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{n-3}$ model, where n is the rank of the GUT group, through the compactification on various orbifolds. The GUT gauge symmetry breaking and doublet–triplet splitting problems have been solved neatly by the orbifold projections, and other interesting phenomenology issues, like μ problems, gauge coupling unifications, non-supersymmetric GUT, gauge–Higgs unification, proton decay, etc., have also been discussed [7,8]. By the way, it seems to us that this approach is similar to the Wilson line approach, but it is not the same; for example, the fundamental group of the extra space manifold can be trivial in the discrete symmetry approach, and we may not break the supersymmetry by a Wilson line approach.

On the other hand, large extra dimension scenarios with branes have been a very interesting subject for the past few years; in those models the gauge hierarchy problem can be solved because the physical volume of extra dimensions may be very large and the higher dimensional Planck scale might be low [9], or the metric for the extra dimensions has a warp factor [10]. Naively, one might think the masses of the KK states are $(\sum_i n_i^2/R_i^2)^{1/2}$, where R_i is the radius of the ith extra dimension. However, it is shown that this is not true if one considers the shape moduli [11] or the local discrete symmetry in the brane neighborhood [12], and it may be possible to maintain the ratio (hierarchy) between the higher dimensional Planck scale and 4-dimensional Planck scale while simultaneously making the KK states arbitrarily heavy. So a lot of experimental bounds on the theories with large extra

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dimensions are relaxed. Moreover, the gauge symmetry and supersymmetry can be broken if we consider the local discrete symmetry [12].

In this paper, we study the supersymmetric GUT models where the supersymmetry and GUT gauge symmetry can be broken by the discrete symmetries in the branes' neighborhoods or on the extra space manifold. We require that for the zero modes in the bulk, the supersymmetric GUT models are broken down to the 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{n-3}$ model, and above the GUT scale or including the zero modes and KK modes, the bulk should preserve the original GUT gauge symmetry and supersymmetries, i.e., we cannot project out all the zero modes and KK modes of the fields in the theories. In addition, we define two discrete symmetries Z_n and Z'_n to be equivalent if
(1) $n = n'$;
(2) in order to satisfy our requirement, the representation for the generator of Z_n in the adjoint representation of the GUT group G must be the same as that for the generator of Z'_n.

First, we would like to explore the general scenarios in which the GUT gauge symmetry and supersymmetry can be broken by the discrete symmetries in the brane neighborhood, and the masses of the KK states can be set arbitrarily heavy. Our ansatz is that there exist discrete symmetries (local or global) in the special branes' neighborhoods, which become the additional constraints on the KK states. The KK states which satisfy the discrete symmetries remain in the theories, while the KK states which do not satisfy the discrete symmetries are projected out. Therefore, we can construct the theories with only zero modes for all the KK modes having been projected out, or the theories which have large extra dimensions and arbitrarily heavy KK modes, because there is no simple relation between the mass scales of the extra dimensions and the masses of the KK states. In addition, the bulk gauge symmetry and supersymmetry can be broken on the special branes for the zero and KK modes, and in the bulk for the zero modes by local and global discrete symmetries.

(I) We generalize our previous models [12] to the models on the space-time $M^4 \times S^1$ and $M^4 \times I^1$, where the M^4 is the 4-dimensional Minkowski space-time. We point out that the general models on $M^4 \times I^1$ can be obtained from the general models on $M^4 \times S^1$ by moduloing the $(Z_2)^{k_0}$ symmetry in which k_0 is a positive integer. Moreover, we find that, to satisfy our ansatz and requirement, there are at most two non-equivalent Z_2 symmetries, which can be local or global. Therefore, we can only discuss the 5-dimensional $N = 1$ supersymmetric $SU(5)$ model. In this scenario, the bulk 4-dimensional $N = 2$ supersymmetry and $SU(5)$ gauge symmetry are broken down to the 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)$ gauge symmetry on the special brane with GUT breaking Z_2 symmetry for all the modes, and in the bulk for the zero modes. The 3-branes preserve half of the bulk supersymmetry. Moreover, the masses of the KK states can be set arbitrarily heavy although the physical size of the fifth dimension can be large, even in the millimeter range. By the way, one can also discuss the non-supersymmetric $SU(6)$ and $SO(10)$ breaking; however, there are zero modes for $A_5^{\hat{a}}$ where $\hat{a}$ is the index related to the broken gauge generators under two non-equivalent Z_2 projections.

(II) We study the models on the space-time $M^4 \times M^1 \times M^1$ where M^1 can be S^1, S^1/Z_2, and I^1. Because the extra space manifold is the product of two 1-dimensional manifold and the discussions are similar, as representatives, we discuss the models on the space-times $M^4 \times S^1 \times S^1$ in which there are parallel 4-branes with Z_2 symmetry along the fifth and sixth dimensions, and there are at most four non-equivalent Z_2 symmetries. We also discuss the models on the space-time $M^4 \times S^1/Z_2 \times S^1/Z_2$ where there are only four 4-branes at the boundaries and some 3-branes in the bulk, and there are three non-equivalent Z_2 symmetries. In those models, the extra dimensions can be large and the masses of the KK states can be set arbitrarily heavy. The 6-dimensional non-supersymmetric GUT models and $N = 1$ supersymmetric GUT models can be considered as special cases of the $N = 2$ supersymmetric GUT models, so we discuss the 6-dimensional $N = 2$ supersymmetric GUT models. Because $N = 2$ 6-dimensional supersymmetric theory has 16 real supercharges, which corresponds to $N = 4$ 4-dimensional supersymmetric theory, we cannot have hypermultiplets in the bulk, and we have to put the standard model fermions on the brane. As representatives, we discuss the 6-dimensional $N = 2$ supersymmetric $SU(6)$ and $SO(10)$ models on the space-time $M^4 \times S^1 \times S^1$, and the 6-dimensional $N = 2$ supersymmetric $SU(6)$ models with gauge–Higgs unification on the space-time $M^4 \times S^1/Z_2 \times S^1/Z_2$. For the zero modes, the bulk 4-dimensional $N = 4$ supersymmetry and $SU(6)$ or $SO(10)$ gauge symmetry are broken down to the 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^2$ gauge symmetry.

Second, we discuss the models where the extra space manifold is the disc D^2 or the annulus A^2. In this kind of scenarios, we can naturally use complex coordinates and introduce global Z_n symmetry for any positive integer n, so we can break any $SU(M)$ gauge symmetry for $M \geq 5$ down to the $SU(3) \times SU(2) \times U(1)^{M-4}$ gauge symmetry. Similar to the above, we only study the 6-dimensional $N = 2$ supersymmetric GUT models with the standard model fermions on the boundary 4-branes or on the 3-brane at the origin if the extra space manifold is the disc D^2. There are 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^{M-4}$ gauge symmetry in the bulk and on the 4-branes for the zero modes, and on the 3-brane at the origin in the disc D^2 scenario. Including all the KK states, we will have 4-dimensional $N = 4$ supersymmetry and $SU(M)$ gauge symmetry in the bulk and on the 4-branes. By the way, if we put the standard model fermions on the 3-brane at the origin, the extra dimensions can be large and the gauge hierarchy problem can be solved, for there does not exist a proton decay problem at all. As an example, we discuss the 6-dimensional $N = 2$ supersymmetric $SU(6)$ model on $M^4 \times A^2$ or $M^4 \times D^2$ with Z_9 symmetry. Moreover, if the extra space manifold is the annulus A^2, for suitable choices of the inner radius and outer radius we might construct the models where only a few KK states are light, and the other KK states are relatively heavy due to the boundary condition on the inner and outer boundaries, so we might produce the light KK states of gauge fields at future colliders, which is very interesting in collider physics.

In addition, if the extra space manifold is a sector of D^2 or a segment of A^2, we point out that the masses of the KK states can be set arbitrarily heavy if the range of the angle is small enough. However, we cannot define the discrete symmetry Z_n for $n > 2$ on a sector of D^2 or a segment of A^2, so it is not interesting for us to discuss the supersymmetric GUT breaking in this case.

Third, we discuss the complete global discrete symmetry on the space-time $M^4 \times T^2$. We prove that the possible global discrete symmetries on the torus is Z_2, Z_3, Z_4, and Z_6. We also discuss the 6-dimensional $N = 2$ supersymmetric $SU(5)$ models on the space-time $M^4 \times T^2$ with Z_6 symmetry, where the standard model fermions on the observable 3-brane at one of the fixed points. There are 4-dimensional $N = 1$ supersymmetry and standard model gauge symmetry in the bulk for the zero modes, and on the 3-brane at the Z_6 fixed point for all the modes. Including the KK states, we will have the 4-dimensional $N = 4$ supersymmetry and $SU(5)$ gauge symmetry in the bulk, the 4-dimensional $N = 1$ supersymmetry and $SU(5)$ gauge symmetry on the 3-branes at the Z_3 fixed points, and the 4-dimensional $N = 4$ supersymmetry and $SU(3) \times SU(2) \times U(1)$ gauge symmetry on the 3-branes at the Z_2 fixed points. The standard model fermions and Higgs fields can be on any 3-brane at one of the fixed points. In particular, if we put the standard model fermions and Higgs fields on the 3-brane at the Z_6 fixed point, the extra dimensions can be large and the gauge hierarchy problem can be solved because there is no proton decay problem at all.

This paper is organized as follows: in Sect. 2 we discuss the discrete symmetry in the brane neighborhood in general. We study the discrete symmetry on the space-time $M^4 \times M^1$, $M^4 \times M^1 \times M^1$, $M^4 \times A^2$, $M^4 \times T^2$ in Sects. 3, 5, 7, 9, respectively. Next, we discuss the supersymmetric GUT breaking on the space-time $M^4 \times M^1$, $M^4 \times M^1 \times M^1$, $M^4 \times A^2$, $M^4 \times T^2$ in Sects. 4, 6, 8, 10, respectively. Our discussion and conclusions are given in Sect. 11.

2 Discrete symmetry in the brane neighborhood

We assume that in a $(4+n)$-dimensional space-time manifold $M^4 \times M^n$ where M^4 is the 4-dimensional Minkowski space-time and M^n is the manifold for extra space dimensions, there exist some topological defects; we call them branes for simplicity. The special branes which we are interested in have co-dimension one or highter. Assuming we have K special branes and using the Ith special brane as a representative, our ansatz is that in the open neighborhood $M^4 \times U_I$ ($U_I \subset M^n$) of the Ith special brane, there is a global or local discete symmetry[1], which forms a discrete group Γ_I, where $I = 1, 2, \cdots, K$. The Lagrangian is invariant under the discrete symmetries. In addition, we require that above the GUT scale or including all the KK states, the bulk should preserve the original GUT gauge symmetries and supersymmetries, i.e., we cannot project out all the KK states of the fields in the theories, and the supersymmetric GUT models are broken down to the 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{n-3}$ model in the bulk for the zero modes.

Assuming that the local coordinates for extra dimensions in the Ith special brane neighborhood are y_I^1, y_I^2, $\cdots$, y_I^n, the action of any element $\gamma_i^I \subset \Gamma_I$ on U_I can be expressed as

$$\begin{aligned}\gamma_i^I: \quad & (y_I^1, y_I^2, \cdots, y_I^n) \subset U_I \\ & \longrightarrow (\gamma_i^I y_I^1, \gamma_i^I y_I^2, \cdots, \gamma_i^I y_I^n) \subset U_I, \end{aligned} \tag{1}$$

where the Ith special brane position is the only fixed point, line, or hypersurface for the whole group Γ_I as long as the neighborhood is small enough.

The Lagrangian is invariant under the discrete symmetry in the neighborhood $M^4 \times U_I$ of the Ith special brane, i.e., for any element $\gamma_i^I \subset \Gamma_I$

$$\mathcal{L}(x^\mu, \gamma_i^I y_I^1, \gamma_i^I y_I^2, \cdots, \gamma_i^I y_I^n) = \mathcal{L}(x^\mu, y_I^1, y_I^2, \cdots, y_I^n), \tag{2}$$

where $(y_I^1, y_I^2, \cdots, y_I^n) \subset U_I$. So, for a generic bulk multiplet Φ which fills a representation of the bulk gauge group G, we have

$$\begin{aligned}&\Phi(x^\mu, \gamma_i^I y_I^1, \gamma_i^I y_I^2, \cdots, \gamma_i^I y_I^n) \\ &= \eta_\Phi^I (R_{\gamma_i^I})^{l_\Phi} \Phi(x^\mu, y_I^1, y_I^2, \cdots, y_I^n)(R_{\gamma_i^I}^{-1})^{m_\Phi}, \end{aligned} \tag{3}$$

where η_Φ^I is an element of the discrete symmetry and can be determined from the Lagrangian (up to an element in Γ_I for the matter fields), l_Φ and m_Φ are the non-negative integers determined by the representation of Φ under the gauge group G. Moreover, $R_{\gamma_i^I}$ is an element in G, and R_{Γ_I} is a discrete subgroup of G. We will choose $R_{\gamma_i^I}$ as the matrix representation for γ_i^I in the adjoint representation of the gauge group G. The consistent condition for $R_{\gamma_i^I}$ is

$$R_{\gamma_i^I} R_{\gamma_j^I} = R_{\gamma_i^I \gamma_j^I}, \quad \forall \gamma_i^I, \ \gamma_j^I \subset \Gamma_I. \tag{4}$$

Mathematical speaking, the map $R: \ \Gamma_I \longrightarrow R_{\Gamma_I} \subset G$ is a homomorphism. Because the special branes are fixed under the discrete symmetry transformations, the gauge group on the Ith special brane is the subgroup of G which commutes with R_{Γ_I}, and we denote the subgroup by G/R_{Γ_I}. For the zero modes, the bulk gauge group is broken down to the subgroup of G which commutes with all

[1] Global discrete symmetry is a "special" case of local discrete symmetry. The key difference is that the space-time manifold can modulo the global discrete symmetry and become a quotient space-time manifold or orbifold

R_{Γ_I}, i.e., R_{Γ_1}, $R_{\Gamma_2}, \cdots, R_{\Gamma_K}$, and we denote the subgroup by $G/\{R_{\Gamma_1}, R_{\Gamma_2}, \cdots, R_{\Gamma_K}\}$. In addition, if the theory is supersymmetric, the special branes will preserve part of the bulk supersymmetry, and the zero modes in the bulk also preserve part of the supersymmetry; in other words, the supersymmetry can be broken on the special branes for all the modes, and in the bulk for the zero modes.

In addition, we only have the KK states which satisfy the local and global discrete symmetries in the theories because the KK modes, which do not satisfy the local and global discrete symmetries, are projected out under our ansatz. Therefore, we can construct the theories with only zero modes because all the KK modes are projected out, or the theories which have large extra dimensions and arbitrarily heavy KK states for there is no simple relation between the mass scales of the extra dimensions and the masses of the KK states. By the way, we are only interested in the second kind of scenarios.

3 Discrete symmetry on the space-time $M^4 \times M^1$

We would like to generalize our previous models [12] to the models on the space-time $M^4 \times S^1$, and $M^4 \times I^1$. We find that the general models on $M^4 \times I^1$ can be obtained from the general models on $M^4 \times S^1$ by moduloing the $(Z_2)^{k_0}$ symmetry in which k_0 is the positive integer. The models on $M^4 \times S^1/(Z_2 \times Z_2')$ [13] are the special case for $k_0 = 2$ and no 3-branes in the bulk.

We assume that the corresponding coordinates for the space-time are x^μ ($\mu = 0, 1, 2, 3$), $y \equiv x^5$, the radius for the circle S^1 is R, and the length for the interval I^1 is πR. We also assume that there are some special 3-branes along the fifth dimension, and there is a Z_2 reflection symmetry in each 3-brane neighborhood.

3.1 Discrete symmetry on $M^4 \times S^1$

Assuming we have $n+1$ parallel 3-branes along the S^1, and their fifth coordinates are $y_0 = 0 < y_1 < y_2 < \cdots < y_n < 2\pi R$, we define the local fifth coordinate for the ith brane $y_i' \equiv y - y_i$, and then, $y_0' \equiv y$. In addition, the equivalence class for the reflection Z_2 symmetry in the ith brane beighborhood is $y_i' \sim -y_i'$. For that Z_2 symmetry, we define the corresponding Z_2 operator P_i for $i = 0, 1, 2, \cdots, n$, whose eigenvalue is ± 1, i.e., for a generic field or function, we have

$$P_i \phi(x^\mu, y_i') = \pm \phi(x^\mu, y_i'). \tag{5}$$

By the way, $P_i^2 = 1$, so $\{1, P_i\}$ forms a Z_2 group. If $y_i/(2\pi R)$ is an irrational number, we will project out all the KK states, which cannot satisfy our requirement. So, we assume

$$y_i = \frac{p_i}{q_i} 2\pi R, \quad \text{for } i = 1, 2, \cdots, n, \tag{6}$$

where p_i and q_i are relative prime positive integers.

Let us assume that L is the least common multiple for all $p_i + q_i$, i.e.,

$$L \equiv [p_1 + q_1, p_2 + q_2, \cdots, p_n + q_n]. \tag{7}$$

By the unique prime factorization theorem, we obtain

$$L = 2^{k_0} s_1^{k_1} s_2^{k_2} \cdots s_j^{k_j}, \tag{8}$$

where $2 < s_1 < s_2 < \cdots < s_j$, k_0 is a non-negative integer, s_i is a prime number and k_i is a positive integer for $i = 1, 2, \cdots, j$. Moreover, we define the effective physical radius r by

$$r = \begin{cases} 4R/L & \text{for } k_0 \geq 2, \\ 2R/L & \text{for } k_0 = 1, \\ R/L & \text{for } k_0 = 0. \end{cases} \tag{9}$$

For a generic bulk field ϕ, we obtain the KK modes expansions

$$\phi_{++}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^{\delta_{n,0}} \pi R}} \phi_{++}^{(2n)}(x^\mu) \cos \frac{2ny}{r}, \tag{10}$$

$$\phi_{+-}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{+-}^{(2n+1)}(x^\mu) \cos \frac{(2n+1)y}{r}, \tag{11}$$

$$\phi_{-+}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{-+}^{(2n+1)}(x^\mu) \sin \frac{(2n+1)y}{r}, \tag{12}$$

$$\phi_{--}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{--}^{(2n+2)}(x^\mu) \sin \frac{(2n+2)y}{r}, \tag{13}$$

where n is a non-negative integer. The 4-dimensional fields $\phi_{++}^{(2n)}$, $\phi_{+-}^{(2n+1)}$, $\phi_{-+}^{(2n+1)}$ and $\phi_{--}^{(2n+2)}$ acquire masses $2n/r$, $(2n+1)/r$, $(2n+1)/r$ and $(2n+2)/r$ upon the compactification. Zero modes are contained only in the ϕ_{++} fields; thus, the matter content of massless sector is smaller than that of the full 5-dimensional multiplet. Moreover, because $0 < r \leq R$, the masses of the KK states (n/r) can be set arbitrarily heavy if L is large enough, i.e., we choose suitable p_i and q_i for some i; for example, if $1/R$ is about TeV, $p_i = 10^{13} - 1$ and $q_i = 10^{13} + 1$, we obtain that $1/r$ is at least about 10^{16} GeV, which is the usual GUT scale. Therefore, there is no simple relation between the physical size of the fifth dimension and the mass scales of the KK modes.

For $k_0 = 0$ and $k_0 = 1$, we obtain

$$P_i \phi_{+\pm}(x^\mu, y) = \phi_{+\pm}(x^\mu, y), \tag{14}$$
$$P_i \phi_{-\pm}(x^\mu, y) = -\phi_{-\pm}(x^\mu, y), \tag{15}$$

for all $i = 0, 1, 2, \cdots, n$. So, we only have one non-equivalent Z_2 symmetry. For $i = 0$, we always have the above equations for P_0.

For $k_0 \geq 2$, if $(p_i + q_i)$ is a multiple of 2^{k_0}, i.e., $2^{k_0} | (p_i + q_i)$, we obtain

$$P_i \phi_{\pm +}(x^\mu, y) = \phi_{\pm +}(x^\mu, y), \tag{16}$$
$$P_i \phi_{\pm -}(x^\mu, y) = -\phi_{\pm -}(x^\mu, y), \tag{17}$$

and if $(p_i + q_i)$ is not a multiple of 2^{k_0}, i.e., $2^{k_0} \not| (p_i + q_i)$, we obtain

$$P_i \phi_{+\pm}(x^\mu, y) = \phi_{+\pm}(x^\mu, y), \tag{18}$$
$$P_i \phi_{-\pm}(x^\mu, y) = -\phi_{-\pm}(x^\mu, y). \tag{19}$$

So we have two non-equivalent Z_2 symmetries.

Because we need a discrete symmetry to break the bulk gauge symmetry and supersymmetry, we will concentrate on the scenario with $k_0 \geq 2$. To be explicit, we would like to give two examples:
(I) $n = 1$ and $4|(p_1 + q_1)$. In this simple case, we can have two local Z_2 symmetries.
(II) Suppose $n = 3$, $4|(p_1 + q_1)$, $p_2 = q_2 = 1$, $p_3 = q_1$, and $q_3 = p_1$, we have one global Z_2 symmetry and one local Z_2 symmetry. The local Z_2 symmetry will become global if $p_1 = 1$ and $q_1 = 3$. This is the scenario discussed in [12] where the global Z_2 symmetry has been moduloed from the manifold.

Furthermore, if we require that the models have one global Z_2 symmetry, then modulo this global Z_2 symmetry we obtain the models with discrete symmetry on the space-time $M^4 \times S^1/Z_2$. The two Z_2 symmetries in the two boundary 3-branes' neighborhoods are equivalent. Let us explain this in detail: suppose we have $2n + 2$ special 3-branes. We require that $y_0 = 0$, $y_{n+1} = \pi R$, $p_i = q_{2n+2-i}$ and $q_i = p_{2n+2-i}$, where $i = 1, 2, \cdots, n$; we will then have one global Z_2 symmetry in which the equivalence class is $y \sim -y$. Moduloing this equivalence class, we obtain the models on $M^4 \times S^1/Z_2$.

In general, if we require that the models have global $(Z_2)^{k_0}$ symmetry for $k_0 > 1$, then modulo the global $(Z_2)^{k_0}$ symmetry, we obtain the models with discrete symmetry on the space-times $M^4 \times S^1/(Z_2)^{k_0}$. The two Z_2 symmetries in the two boundary 3-branes' neighborhoods are not equivalent. As an example, we discuss the models with $k_0 = 2$. Suppose we have $4n + 4$ 3-branes, $y_{n+1} = \pi R/2$, $y_{2n+2} = \pi R$, $y_{3n+3} = 3\pi R/2$. For $i = 1, 2, \cdots, n$, we have $p_i = q_{4n+4-i}$, $q_i = p_{4n+4-i}$, $p_{2n+2-i} = p'_i$, $q_{2n+2-i} = q'_i$, $p_{2n+2+i} = q'_i$, $q_{2n+2+i} = p'_i$, where p'_i and q'_i are relative prime positive integer and satisfy the equation

$$\frac{p'_i}{q'_i} = \frac{q_i - p_i}{2(p_i + q_i)}. \tag{20}$$

If $n = 0$, we obtain the models on $M^4 \times S^1/(Z_2 \times Z'_2)$ [13].

3.2 Discrete symmetry on $M^4 \times I^1$

In this subsection, we would like to consider the discrete symmetry on the space-time $M^4 \times I^1$. Assuming that on two boundary 3-branes, the fields should satisfy the Dirichlet or Neumann boundary condition, we show that the general models on the space-time $M^4 \times I^1$ contain the models on $M^4 \times S^1/Z_2$ and the models on $M^4 \times S^1/(Z_2)^{k_0}$ for $k_0 > 1$. We assume that we have $n + 2$ parallel 3-branes along the I^1, and that their fifth coordinates are $y_0 = 0 < y_1 < y_2 < \cdots < y_{n+1} = \pi R$. We define the local fifth coordinate for the ith brane by $y'_i \equiv y - y_i$. The equivalent class for the reflection Z_2 symmetry in the ith brane neighborhood is $y'_i \sim -y'_i$. Moreover, for that Z_2 symmetry, we define the corresponding Z_2 operator P_i for $i = 0, 1, 2, \cdots, n$, whose eigenvalue is ± 1; i.e., for a generic field or function, we have

$$P_i \phi(x^\mu, y'_i) = \pm \phi(x^\mu, y'_i). \tag{21}$$

Because if $y_i/(\pi R)$ is an irrational number we will project out all the KK states, which cannot satisfy our requirement. So, we assume

$$y_i = \frac{p_i}{q_i} \pi R, \quad \text{for } i = 1, 2, \cdots, n, \tag{22}$$

where p_i and q_i are relative prime positive integers.

Assume L is the least common multiple for all $p_i + q_i$, i.e.,

$$L \equiv [p_1 + q_1, p_2 + q_2, \cdots, p_n + q_n]. \tag{23}$$

By the unique prime factorization theorem, we obtain

$$L = 2^{l_0} s_1^{l_1} s_2^{l_2} \cdots s_j^{l_j}, \tag{24}$$

where $2 < s_1 < s_2 < \cdots < s_j$, l_0 is a non-negative integer, s_i is a prime number and l_i is a positive integer for $i = 1, 2, \cdots, j$. For the models on the space-time $M^4 \times S^1/Z_2$, we define the effective physical radius r by

$$r = \begin{cases} 2R/L & \text{for } l_0 \geq 1, \\ R/L & \text{for } l_0 = 0. \end{cases} \tag{25}$$

For $l_0 = 0$, we can still define the effective radius by

$$r = \frac{2R}{L}. \tag{26}$$

This kind of models cannot be obtained from the models in the last subsection by moduloing one global Z_2 symmetry; however, they can be obtained from the models in the last subesection by moduloing global $(Z_2)^{k_0}$ symmetries for $k_0 > 1$.

For a generic bulk field ϕ, we obtain the KK modes expansions

$$\phi_{++}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^{\delta_{n,0}} \pi R}} \phi_{++}^{(2n)}(x^\mu) \cos \frac{2ny}{r}, \tag{27}$$

$$\phi_{+-}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{+-}^{(2n+1)}(x^\mu) \cos \frac{(2n+1)y}{r}, \tag{28}$$

$$\phi_{-+}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{-+}^{(2n+1)}(x^\mu) \sin \frac{(2n+1)y}{r}, \tag{29}$$

$$\phi_{--}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{--}^{(2n+2)}(x^\mu) \sin \frac{(2n+2)y}{r}, \tag{30}$$

where n is a non-negative integer. The 4-dimensional fields $\phi_{++}^{(2n)}$, $\phi_{+-}^{(2n+1)}$, $\phi_{-+}^{(2n+1)}$ and $\phi_{--}^{(2n+2)}$ acquire masses $2n/r$, $(2n+1)/r$, $(2n+1)/r$ and $(2n+2)/r$ upon the compactification. The zero modes are contained only in the ϕ_{++}

fields. Moreover, because $0 < r \leq R$, the masses of the KK states (n/r) can be set arbitrarily heavy if L is large enough, i.e., we choose suitable p_i and q_i, for some i. So there is no simple relation between the physical size of the fifth dimension and the mass scales of KK states.

(I) First, we discuss the models on the space-times $M^4 \times S^1/Z_2$. We should keep in mind that there is one global Z_2 symmetry that has been moduloed from S^1, and which we can call P_0 or P_{n+1}.

For $l_0 = 0$, we obtain

$$P_i\phi_{+\pm}(x^\mu, y) = \phi_{+\pm}(x^\mu, y), \quad (31)$$

$$P_i\phi_{-\pm}(x^\mu, y) = -\phi_{-\pm}(x^\mu, y), \quad (32)$$

for all $i = 1, 2, \cdots, n$. Under our assumption, these Z_2 symmetries are equivalent to the global Z_2 symmetry, so we just have one independent Z_2 symmetry.

For $l_0 \geq 1$, if $(p_i + q_i)$ is a multiple of 2^{l_0}, i.e., $2^{l_0}|(p_i + q_i)$, we obtain

$$P_i\phi_{\pm+}(x^\mu, y) = \phi_{\pm+}(x^\mu, y), \quad (33)$$

$$P_i\phi_{\pm-}(x^\mu, y) = -\phi_{\pm-}(x^\mu, y); \quad (34)$$

this Z_2 symmetry is not equivalent to the global symmetry. If $(p_i + q_i)$ is not a multiple of 2^{l_0}, i.e., $2^{l_0} \not| (p_i + q_i)$, we obtain

$$P_i\phi_{+\pm}(x^\mu, y) = \phi_{+\pm}(x^\mu, y), \quad (35)$$

$$P_i\phi_{-\pm}(x^\mu, y) = -\phi_{-\pm}(x^\mu, y), \quad (36)$$

this Z_2 symmetry is equivalent to the global Z_2 symmetry. In short, we have two independent Z_2 symmetries, in which one can be considered as a global Z_2 symmetry.

Because we need a discrete symmetry to break the bulk gauge symmetry and supersymmetry, we will concentrate on the scenario with $l_0 \geq 1$. To be explicit, we would like to give one example: $n = 1$ and $2|(p_1 + q_1)$. In this simple case, we can have one local Z_2 symmetries and one global Z_2 symmetry. That local Z_2 symmetry becomes global if $p_1 = q_1 = 1$.

(II) If $l_0 = 0$ and $r = 2R/L$, this kind of models can be obtained from the models in the last subsection by moduloing the global $(Z_2)^{k_0}$ symmetries for $k_0 > 1$, so, P_0 and P_{n+1} are two non-equivalent Z_2 symmetries. If at the starting point, the original extra space manifold is I^1, we can consider that there are no global Z_2 symmetries for P_0 and P_{n+1}.

Because $l_0 = 0$, there are two cases: p_i is odd and q_i is even, or p_i is even and q_i is odd. If p_i is even,

$$P_i\phi_{+\pm}(x^\mu, y) = \phi_{+\pm}(x^\mu, y), \quad (37)$$

$$P_i\phi_{-\pm}(x^\mu, y) = -\phi_{-\pm}(x^\mu, y), \quad (38)$$

and if p_i is odd

$$P_i\phi_{\pm+}(x^\mu, y) = \phi_{\pm+}(x^\mu, y), \quad (39)$$

$$P_i\phi_{\pm-}(x^\mu, y) = -\phi_{\pm-}(x^\mu, y). \quad (40)$$

Therefore, we can have two local Z_2 symmetries, which can be thought of as global symmetries if we consider the original manifold for the extra dimension to be S^1.

4 GUT breaking on the space-time $M^4 \times M^1$

In this section, we would like to discuss the supersymmetric $SU(5)$ model on the space-time $M^4 \times M^1$ with two discrete Z_2 symmetries. We assume that there are $SU(5)$ gauge fields and two 5-plet Higgs hypermultiplets in the bulk, and the standard model fermions can be on the 3-brane or in the bulk.

As we know, the $N = 1$ supersymmetric theory in 5-dimension have 8 real supercharges, corresponding to $N = 2$ supersymmetry in 4-dimension. The vector multiplet physically contains a vector boson A_M where $M = 0, 1, 2, 3, 5$, two Weyl gauginos $\lambda_{1,2}$, and a real scalar σ. In terms of 4-dimensional $N = 1$ language, it contains a vector multiplet $V(A_\mu, \lambda_1)$ and a chiral multiplet $\Sigma((\sigma + \mathrm{i}A_5)/2^{1/2}, \lambda_2)$ which transform in the adjoint representation of $SU(5)$. And the 5-dimensional hypermultiplet physically has two complex scalars ϕ and ϕ^{c}, a Dirac fermion Ψ, and can be decomposed into two 4-dimensional chiral multiplets $\Phi(\phi, \psi \equiv \Psi_{\mathrm{R}})$ and $\Phi^{\mathrm{c}}(\phi^{\mathrm{c}}, \psi^{\mathrm{c}} \equiv \Psi_{\mathrm{L}})$, which transform as conjugate representations of each other under the gauge group. For instance, we have two Higgs chiral multiplets H_u and H_d, which transform as 5 and $\bar{5}$ under the $SU(5)$ gauge symmetry, and their mirror H_u^{c} and H_d^{c}, which transform as $\bar{5}$ and 5 under the $SU(5)$ gauge symmetry.

The general action for the $SU(5)$ gauge fields and their couplings to the bulk hypermultiplet Φ is [14]

$$\begin{aligned} S = &\int \mathrm{d}^5x \frac{1}{kg^2}\mathrm{Tr}\Bigg[\frac{1}{4}\int \mathrm{d}^2\theta\,(W^\alpha W_\alpha + \mathrm{H.C.}) \\ &+ \int \mathrm{d}^4\theta\Big((\sqrt{2}\partial_5 + \bar{\Sigma})\mathrm{e}^{-V}(-\sqrt{2}\partial_5 + \Sigma)\mathrm{e}^V \\ &+ \partial_5 \mathrm{e}^{-V}\partial_5 \mathrm{e}^V\Big)\Bigg] \\ &+ \int \mathrm{d}^5x\left[\int \mathrm{d}^4\theta\,(\Phi^{\mathrm{c}}\mathrm{e}^V\bar{\Phi}^{\mathrm{c}} + \bar{\Phi}\mathrm{e}^{-V}\Phi)\right. \\ &\left.+ \int \mathrm{d}^2\theta\left(\Phi^{\mathrm{c}}\left(\partial_5 - \frac{1}{\sqrt{2}}\Sigma\right)\Phi + \mathrm{H.C.}\right)\right]. \end{aligned} \quad (41)$$

Because the action is invariant under the parity P_i, we find that under the parity operator P_i the vector multiplet transforms as

$$V(x^\mu, y_i') \to V(x^\mu, -y_i') = P_i V(x^\mu, y_i') P_i^{-1}, \quad (42)$$

$$\Sigma(x^\mu, y_i') \to \Sigma(x^\mu, -y_i') = -P_i \Sigma(x^\mu, y_i') P_i^{-1}; \quad (43)$$

if the hypermultiplet Φ is a 5 or $\bar{5}$ $SU(5)$ multiplet, we have

$$\Phi(x^\mu, y_i') \to \Phi(x^\mu, -y_i') = \eta_\Phi P_i \Phi(x^\mu, y_i'), \quad (44)$$

$$\Phi^{\mathrm{c}}(x^\mu, y_i') \to \Phi^{\mathrm{c}}(x^\mu, -y_i') = -\eta_\Phi P_i \Phi^{\mathrm{c}}(x^\mu, y_i'), \quad (45)$$

and if the hypermultiplet Φ is a 10 or $\bar{10}$ $SU(5)$ multiplet, we have

$$\Phi(x^\mu, y_i') \to \Phi(x^\mu, -y_i') = \eta_\Phi P_i \Phi(x^\mu, y_i') P_i^{-1}, \quad (46)$$

$$\Phi^{\mathrm{c}}(x^\mu, y_i') \to \Phi^{\mathrm{c}}(x^\mu, -y_i') = -\eta_\Phi P_i \Phi^{\mathrm{c}}(x^\mu, y_i') P_i^{-1}, \quad (47)$$

Table 1. Parity assignment and masses ($n \geq 0$) of the fields in the $SU(5)$ gauge and Higgs multiplets. The indices F, T are for doublet and triplet, respectively

(P, P')	Field	Mass
$(+,+)$	V_μ^a, H_u^F, H_d^F	$\frac{2n}{r}$
$(+,-)$	$V_\mu^{\hat{a}}$, H_u^T, H_d^T	$\frac{2n+1}{r}$
$(-,+)$	$\Sigma^{\hat{a}}$, H_u^{cT}, H_d^{cT}	$\frac{2n+1}{r}$
$(-,-)$	Σ^a, H_u^{cF}, H_d^{cF}	$\frac{2n+2}{r}$

where $\eta_\Phi = \pm 1$.

For simplicity, let us denote the two non-equivalent Z_2 symmetries by P and P'. We choose the following matrix representations for the parities P and P' which are expressed in the adjoint representaion of $SU(5)$

$$P = \mathrm{diag}(+1, +1, +1, +1, +1),$$
$$P' = \mathrm{diag}(-1, -1, -1, +1, +1). \quad (48)$$

So, upon using the parity P', the gauge generators T^A where $A = 1, 2, \cdots, 24$ for $SU(5)$ are separated into two sets: T^a are the gauge generators for the standard model gauge group, and $T^{\hat{a}}$ are the other broken gauge generators

$$PT^aP^{-1} = T^a, \quad PT^{\hat{a}}P^{-1} = T^{\hat{a}}, \quad (49)$$

$$P'\, T^aP'^{-1} = T^a, \quad P'\, T^{\hat{a}}P'^{-1} = -T^{\hat{a}}. \quad (50)$$

Choosing $\eta_{H_u} = +1$ and $\eta_{H_d} = +1$, we obtain the particle spectra, which are given in Table 1. The bulk 4-dimensional $N = 2$ supersymmetry and $SU(5)$ gauge symmetry are broken down to the 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)$ gauge symmetry in the bulk for the zero modes, and on the special 3-branes which preserve Z_2 symmetry P' for all the modes. Including the KK states, the gauge symmetry on the special 3-branes which preserve the Z_2 symmetry P, is $SU(5)$. In addition, the 4-dimensional supersymmetry on the 3-branes is $1/2$ of the bulk 4-dimensional supersymmetry or $N = 1$ due to the Z_2 symmetry in the brane neighborhood. Moreover, the standard model fermions can be in the bulk or on the 3-brane, and the discussions are similar to those in [4–8], so we will not repeat them here.

By the way, one can also discuss the non-supersymmetric $SU(6)$ and $SO(10)$ breaking; however, there are zero modes for $A_5^{\hat{a}}$ where $\hat{a}$ is the index related to the broken gauge generators under two Z_2 symmetries.

5 Discrete symmetry on the space-time $M^4 \times M^1 \times M^1$

We would like to discuss the models where there are some parallel 4-branes with Z_2 reflection symmetry along the fifth and sixth dimensions on the space-time $M^4 \times M^1 \times M^1$, in which M^1 can be S^1, S^1/Z_2, and I^1. Because the extra space manifold is the product of two 1-dimensional manifolds, we only discuss the models on the space-times $M^4 \times S^1 \times S^1$ as a representative because the discussions of the models with other combinations are similar. In addition, we discuss the models on the space-time $M^4 \times S^1/Z_2 \times S^1/Z_2$ where there are some 3-branes with Z_2 symmetry in the bulk.

The corresponding coordinates for the space-time are x^μ, $(\mu = 0, 1, 2, 3)$ $y \equiv x^5$, $z \equiv x^6$, and the radii for the y and z directions are R_1 and R_2, respectively.

5.1 Discrete symmetry on $M^4 \times S^1 \times S^1$

First, we would like to discuss the discrete symmetry on the space-time $M^4 \times S^1 \times S^1$. Assume that along the y and z directions, we have $n + 1$ and $m + 1$ parallel 4-branes with Z_2 reflection symmetry, respectively. The 5th coordinates for the parallel 4-branes along the y direction are $y_0 = 0 < y_1 < y_2 < \cdots < y_n < 2\pi R_1$, and the 6th coordinates for the parallel 4-branes along the z direction are $z_0 = 0 < z_1 < z_2 < \cdots < z_m < 2\pi R_2$.

We denote the local coordinate for the ith 4-brane along the y direction by $y'_i \equiv y - y_i$. In addition, for the Z_2 symmetry in the ith 4-brane neighborhood, we define a Z_2 operator P_i^y for $i = 0, 1, 2, \cdots, n$, whose eigenvalue is ± 1, i.e., for a generic field or function, we have

$$P_i^y \phi(x^\mu, y'_i, z) = \pm \phi(x^\mu, y'_i, z). \quad (51)$$

Similarly, we denote the local coordinate for the ith 4-brane along the z direction by $z'_i \equiv z - z_i$. For the Z_2 symmetry in the ith 4-brane neighborhood, we define a Z_2 operator P_i^z for $i = 0, 1, 2, \cdots, m$:

$$P_i^z \phi(x^\mu, y, z'_i) = \pm \phi(x^\mu, y, z'_i). \quad (52)$$

Because if $y_i/(2\pi R_1)$ or $z_i/(2\pi R_1)$ is an irrational number, we will project out all the KK states, which cannot satisfy our requirement. So we assume

$$y_i = \frac{p_i^y}{q_i^y} 2\pi R_1, \quad \text{for } i = 1, 2, \cdots, n, \quad (53)$$

$$z_i = \frac{p_i^z}{q_i^z} 2\pi R_2, \quad \text{for } i = 1, 2, \cdots, m, \quad (54)$$

where p_i^y and q_i^y are relative prime positive integers, and p_i^z and q_i^z are relative prime positive integers.

Let us assume that L_y is the least common multiple for all $p_i^y + q_i^y$, and L_z is the least common multiple for all $p_i^z + q_i^z$, i.e.,

$$L_y \equiv [p_1^y + q_1^y, p_2^y + q_2^y, \cdots, p_n^y + q_n^y], \quad (55)$$
$$L_z \equiv [p_1^z + q_1^z, p_2^z + q_2^z, \cdots, p_m^z + q_m^z]. \quad (56)$$

By the unique prime factorization theorem, we find that

$$L_y = 2^{k_0} s_1^{k_1} s_2^{k_2} \cdots s_u^{k_u}, \quad (57)$$
$$L_z = 2^{l_0} t_1^{l_1} t_2^{l_2} \cdots t_v^{l_v}, \quad (58)$$

where $2 < s_1 < s_2 < \cdots < s_u$, $2 < t_1 < t_2 < \cdots < t_v$, k_0 and l_0 are non-negative integers, s_i and t_j are prime numbers, and k_i and l_j are positive integers for $i = 1, 2, \cdots, u$

and $j = 1, 2, \cdots, v$. We define the effective physical radii r_1 and r_2 by

$$r_1 = \begin{cases} 4R_1/L_y & \text{for } k_0 \geq 2, \\ 2R_1/L_y & \text{for } k_0 = 1, \\ R_1/L_y & \text{for } k_0 = 0, \end{cases} \tag{59}$$

$$r_2 = \begin{cases} 4R_1/L_z & \text{for } l_0 \geq 2, \\ 2R_1/L_z & \text{for } l_0 = 1, \\ R_1/L_z & \text{for } l_0 = 0. \end{cases} \tag{60}$$

For a generic bulk field ϕ, we obtain the KK modes expansions

$$\phi_{++++}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{++++}^{(2n,2m)}(x^\mu) A_{++}^{2n}(y, r_1) A_{++}^{2m}(z, r_2), \tag{61}$$

$$\phi_{+++-}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{+++-}^{(2n,2m+1)}(x^\mu) A_{++}^{2n}(y, r_1) A_{+-}^{2m+1}(z, r_2), \tag{62}$$

$$\phi_{++-+}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{++-+}^{(2n,2m+1)}(x^\mu) A_{++}^{2n}(y, r_1) A_{-+}^{2m+1}(z, r_2), \tag{63}$$

$$\phi_{++--}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{++--}^{(2n,2m+2)}(x^\mu) A_{++}^{2n}(y, r_1) A_{--}^{2m+2}(z, r_2), \tag{64}$$

$$\phi_{+-++}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{+-++}^{(2n+1,2m)}(x^\mu) A_{+-}^{2n+1}(y, r_1) A_{++}^{2m}(z, r_2), \tag{65}$$

$$\phi_{+-+-}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{+-+-}^{(2n+1,2m+1)}(x^\mu) A_{+-}^{2n+1}(y, r_1) A_{+-}^{2m+1}(z, r_2), \tag{66}$$

$$\phi_{+--+}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{+--+}^{(2n+1,2m+1)}(x^\mu) A_{+-}^{2n+1}(y, r_1) A_{-+}^{2m+1}(z, r_2), \tag{67}$$

$$\phi_{+---}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{+---}^{(2n+1,2m+2)}(x^\mu) A_{+-}^{2n+1}(y, r_1) A_{--}^{2m+2}(z, r_2), \tag{68}$$

$$\phi_{-+++}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{-+++}^{(2n+1,2m)}(x^\mu) A_{-+}^{2n+1}(y, r_1) A_{++}^{2m}(z, r_2), \tag{69}$$

$$\phi_{-++-}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{-++-}^{(2n+1,2m+1)}(x^\mu) A_{-+}^{2n+1}(y, r_1) A_{+-}^{2m+1}(z, r_2), \tag{70}$$

$$\phi_{-+-+}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{-+-+}^{(2n+1,2m+1)}(x^\mu) A_{-+}^{2n+1}(y, r_1) A_{-+}^{2m+1}(z, r_2), \tag{71}$$

$$\phi_{-+--}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{-+--}^{(2n+1,2m+2)}(x^\mu) A_{-+}^{2n+1}(y, r_1) A_{--}^{2m+2}(z, r_2), \tag{72}$$

$$\phi_{--++}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{--++}^{(2n+2,2m)}(x^\mu) A_{--}^{2n+2}(y, r_1) A_{++}^{2m}(z, r_2), \tag{73}$$

$$\phi_{--+-}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{--+-}^{(2n+2,2m+1)}(x^\mu) A_{--}^{2n+2}(y, r_1) A_{+-}^{2m+1}(z, r_2), \tag{74}$$

$$\phi_{---+}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{---+}^{(2n+2,2m+1)}(x^\mu) A_{--}^{2n+2}(y, r_1) A_{-+}^{2m+1}(z, r_2), \tag{75}$$

$$\phi_{----}(x^\mu, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \phi_{----}^{(2n+2,2m+2)}(x^\mu) A_{--}^{2n+2}(y, r_1) A_{--}^{2m+2}(z, r_2), \tag{76}$$

where

$$A_{++}^{2n}(y, r_1) = \frac{1}{\sqrt{2^{\delta_{n,0}}\pi R_1}} \cos\frac{2ny}{r_1}, \tag{77}$$

$$A_{+-}^{2n+1}(y, r_1) = \frac{1}{\sqrt{\pi R_1}} \cos\frac{(2n+1)y}{r_1}, \tag{78}$$

$$A_{-+}^{2n+1}(y, r_1) = \frac{1}{\sqrt{\pi R_1}} \sin\frac{(2n+1)y}{r_1}, \tag{79}$$

$$A_{--}^{2n+2}(y, r_1) = \frac{1}{\sqrt{\pi R_1}} \sin\frac{(2n+2)y}{r_1}. \tag{80}$$

Similarly, we define $A_{++}^{2n}(z, r_2)$, $A_{+-}^{2n+1}(z, r_2)$, $A_{-+}^{2n+1}(z, r_2)$, $A_{--}^{2n+2}(z, r_2)$.

The 4-dimensional fields $\phi^{(n,m)}$ acquire masses $(n^2/r_1^2 + m^2/r_2^2)^{1/2}$ upon the compactification. The zero modes are contained only in the ϕ_{++++} fields. Moreover, because $0 < r_1 \leq R_1$ and $0 < r_2 \leq R_2$, the masses of the KK states $((n^2/r_1^2 + m^2/r_2^2)^{1/2})$ can be set arbitrarily heavy if L_y and L_z are large enough, i.e., we choose suitable (p_i^y, q_i^y) for some i, and (p_j^z, q_j^z) for some j. So there is no simple relation between the physical size of the extra dimensions and the mass scales of KK modes.

For $k_0 = 0$ and $k_0 = 1$, we obtain

$$P_i^y \phi_{+\pm\pm\pm}(x^\mu, y, z) = \phi_{+\pm\pm\pm}(x^\mu, y, z), \tag{81}$$

$$P_i^y \phi_{-\pm\pm\pm}(x^\mu, y, z) = -\phi_{-\pm\pm\pm}(x^\mu, y, z), \tag{82}$$

for all $i = 0, 1, 2, \cdots, n$. So we only have one independent Z_2 symmetry. For $i = 0$, we always have the above equations for P_0^y.

Moreover, for $k_0 \geq 2$, if $(p_i^y + q_i^y)$ is a multiple of 2^{k_0}, i.e., $2^{k_0}|(p_i^y + q_i^y)$, we obtain

$$P_i^y \phi_{\pm+\pm\pm}(x^\mu, y, z) = \phi_{\pm+\pm\pm}(x^\mu, y, z), \tag{83}$$

$$P_i^y \phi_{\pm-\pm\pm}(x^\mu, y, z) = -\phi_{\pm-\pm\pm}(x^\mu, y, z), \tag{84}$$

and if $(p_i^y+q_i^y)$ is not a multiple of 2^{k_0}, i.e., $2^{k_0} \not| (p_i^y+q_i^y)$, we obtain

$$P_i^y \phi_{+\pm\pm\pm}(x^\mu, y, z) = \phi_{+\pm\pm\pm}(x^\mu, y, z), \quad (85)$$
$$P_i^y \phi_{-\pm\pm\pm}(x^\mu, y, z) = -\phi_{-\pm\pm\pm}(x^\mu, y, z). \quad (86)$$

So we have two non-equivalent Z_2 symmetries along the y direction.

Similarly, for $l_0 = 0$ and $l_0 = 1$, we obtain

$$P_i^z \phi_{\pm\pm+\pm}(x^\mu, y, z) = \phi_{\pm\pm+\pm}(x^\mu, y, z), \quad (87)$$
$$P_i^z \phi_{\pm\pm-\pm}(x^\mu, y, z) = -\phi_{\pm\pm-\pm}(x^\mu, y, z), \quad (88)$$

for all $i = 0, 1, 2, \cdots, m$. So we only have one independent Z_2 symmetry. For $i = 0$, we always have the above equations for P_0^z.

For $l_0 \geq 2$, if $(p_i^z+q_i^z)$ is a multiple of 2^{l_0}, i.e., $2^{l_0}|(p_i^z+q_i^z)$, we obtain

$$P_i^z \phi_{\pm\pm\pm+}(x^\mu, y, z) = \phi_{\pm\pm\pm+}(x^\mu, y, z), \quad (89)$$
$$P_i^z \phi_{\pm\pm\pm-}(x^\mu, y, z) = -\phi_{\pm\pm\pm-}(x^\mu, y, z), \quad (90)$$

and if $(p_i^z+q_i^z)$ is not a multiple of 2^{l_0}, i.e., $2^{l_0} \not| (p_i^z+q_i^z)$ we obtain

$$P_i^z \phi_{\pm\pm+\pm}(x^\mu, y, z) = \phi_{\pm\pm+\pm}(x^\mu, y, z), \quad (91)$$
$$P_i^z \phi_{\pm\pm-\pm}(x^\mu, y, z) = -\phi_{\pm\pm-\pm}(x^\mu, y, z). \quad (92)$$

So we have two non-equivalent Z_2 symmetries along the z direction.

Therefore, we can have at most four non-equivalent Z_2 symmetries. Because we need a discrete symmetry to break the bulk gauge symmetry and supersymmetry, we will concentrate on the scenario with $k_0 \geq 2$ and $l_0 \geq 2$. To be explicit, we would like to give two examples.

(I) $n = 1$, $m = 1$, $4|(p_1^y+q_1^y)$, $4|(p_1^z+q_1^z)$. In this simple case, we can have four local Z_2 symmetries.

(II) Suppose $n = 3$, $m = 3$, $4|(p_1^y+q_1^y)$, $4|(p_1^z+q_1^z)$, $p_2^y = q_2^y = 1$, $p_2^z = q_2^z = 1$, $p_3^y = q_1^y$, $q_3^y = p_1^y$, $p_3^z = q_1^z$, and $q_3^z = p_1^z$. Now we will have two global Z_2 symmetries and two local Z_2 symmetries. The local Z_2 symmetry will become global if $p_1^y = 1$ and $q_1^y = 3$, or $p_1^z = 1$ and $q_1^z = 3$. The two global Z_2 symmetries can be moduloed from the manifold. In general, we may have global $(Z_2)^{k_0}$ and $(Z_2)^{l_0}$ symmetries; the extra space orbifold will be $S^1/(Z_2)^{k_0} \times S^1/(Z_2)^{l_0}$ if we modulo those global Z_2 symmetries.

5.2 Discrete symmetry on $M^4 \times S^1/Z_2 \times S^1/Z_2$

In this subsection, we would like to discuss the discrete symmetry on the space-time $M^4 \times S^1/Z_2 \times S^1/Z_2$, where there are some 3-branes with Z_2 symmetry in the bulk. And we denote two global Z_2 symmtries $y \sim -y$ and $z \sim -z$ as P^y and P^z. For simplicity, we assume that there are only four 4-branes which are the boundary branes on $S^1/Z_2 \times S^1/Z_2$, and the 3-branes are only in the bulk.

Suppose we have n 3-branes in the bulk, and their coodinates are (y_i, z_i) where $0 < y_i < \pi R_1$ and $0 < z_i < \pi R_2$. We denote the local coordinates for the ith 3-brane by $y_i' \equiv y - y_i$, and $z_i' \equiv z - z_i$. Then the equivalent class for the reflection Z_2 symmetry in the ith 3-brane neighborhood is $(y_i', z_i') \sim (-y_i', -z_i')$. For that Z_2 symmetry, we define the corresponding Z_2 operator P_i for $i = 1, 2, \cdots, n$, whose eigenvalue is ± 1, i.e., for a generic field or function; we have

$$P_i \phi(x^\mu, -y_i', -z_i') = \pm \phi(x^\mu, y_i', z_i'). \quad (93)$$

Because if $y_i/(2\pi R_1)$ or $z_i/(2\pi R_1)$ is an irrational number, we will project out all the KK states, which cannot satisfy our requirement. So we assume

$$y_i = \frac{p_i^y}{q_i^y} \pi R_1, \quad \text{for } i = 1, 2, \cdots, n, \quad (94)$$
$$z_i = \frac{p_i^z}{q_i^z} \pi R_2, \quad \text{for } i = 1, 2, \cdots, m, \quad (95)$$

where p_i^y and q_i^y are relative prime positive integers, and p_i^z and q_i^z are relative prime positive integers.

Assume L_y is the least common multiple for all $p_i^y+q_i^y$, and L_z is the least common multiple for all $p_i^z+q_i^z$, i.e.,

$$L_y \equiv [p_1^y+q_1^y, p_2^y+q_2^y, \cdots, p_n^y+q_n^y], \quad (96)$$
$$L_z \equiv [p_1^z+q_1^z, p_2^z+q_2^z, \cdots, p_m^z+q_m^z]. \quad (97)$$

By the unique prime factorization theorem, we obtain

$$L_y = 2^{k_0} s_1^{k_1} s_2^{k_2} \cdots s_u^{k_u}, \quad (98)$$
$$L_z = 2^{l_0} t_1^{l_1} t_2^{l_2} \cdots t_v^{l_v}, \quad (99)$$

where $2 < s_1 < s_2 < \cdots < s_u$, $2 < t_1 < t_2 < \cdots < t_v$, k_0 and l_0 are non-negative integers, s_i and t_j are prime numbers, and k_i and l_j are positive integers for $i = 1, 2, \cdots, u$ and $j = 1, 2, \cdots, v$. We define the effective physical radii r_1 and r_2 by

$$r_1 = \begin{cases} 2R_1/L_y & \text{for } k_0 \geq 1, \\ R_1/L_y & \text{for } k_0 = 0, \end{cases} \quad (100)$$

$$r_2 = \begin{cases} 2R_2/L_z & \text{for } l_0 \geq 1, \\ R_2/L_z & \text{for } l_0 = 0. \end{cases} \quad (101)$$

Because we require that all fields have zero modes or KK modes, we can only have one additional non-equivalent Z_2 symmetry for all the 3-branes, which is not equivalent to two global Z_2 symmetries P^y and P^z. In this scenario, $k_0 \geq 1$, $l_0 \geq 1$; $2^{k_0}|(p_i^y+q_i^y)$ for $i = 1, 2, \cdots, n$, $2^{l_0}|(p_j^z+q_j^z)$ for $j = 1, 2, \cdots, m$.

Because all the Z_2 symmetries for the 3-branes are equivalent, we write them as $P^{y'z'}$. Denoting the field with $(P^y, P^z, P^{y'z'}) = (\pm, \pm, \pm)$ by $\phi_{\pm\pm\pm}$, we obtain the following KK mode expansions:

$$\phi_{+++}(x^\mu, y, z)$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Big(\phi_{+++}^{(2n,2m)}(x^\mu) A_{++}^{2n}(y, r_1) A_{++}^{2m}(z, r_2)$$

$$+ \phi_{+++}^{(2n+1,2m+1)}(x^\mu) A_{+-}^{2n+1}(y,r_1) A_{+-}^{2m+1}(z,r_2)\Big), \quad (102)$$

$$\phi_{++-}(x^\mu,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big(\phi_{++-}^{(2n,2m+1)}(x^\mu) A_{++}^{2n}(y,r_1) A_{+-}^{2m+1}(z,r_2) + \phi_{++-}^{(2n+1,2m)}(x^\mu) A_{+-}^{2n+1}(y,r_1) A_{++}^{2m}(z,r_2)\Big), \quad (103)$$

$$\phi_{+-+}(x^\mu,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big(\phi_{+-+}^{(2n,2m+1)}(x^\mu) A_{++}^{2n}(y,r_1) A_{-+}^{2m+1}(z,r_2) + \phi_{+-+}^{(2n+1,2m+2)}(x^\mu) A_{+-}^{2n+1}(y,r_1) A_{--}^{2m+2}(z,r_2)\Big), \quad (104)$$

$$\phi_{+--}(x^\mu,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big(\phi_{+--}^{(2n,2m+2)}(x^\mu) A_{++}^{2n}(y,r_1) A_{--}^{2m+2}(z,r_2) + \phi_{+--}^{(2n+1,2m+1)}(x^\mu) A_{+-}^{2n+1}(y,r_1) A_{-+}^{2m+1}(z,r_2)\Big), \quad (105)$$

$$\phi_{-++}(x^\mu,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big(\phi_{-++}^{(2n+1,2m)}(x^\mu) A_{-+}^{2n+1}(y,r_1) A_{++}^{2m}(z,r_2) + \phi_{-++}^{(2n+2,2m+1)}(x^\mu) A_{--}^{2n+2}(y,r_1) A_{+-}^{2m+1}(z,r_2)\Big), \quad (106)$$

$$\phi_{-+-}(x^\mu,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big(\phi_{-+-}^{(2n+1,2m+1)}(x^\mu) A_{-+}^{2n+1}(y,r_1) A_{+-}^{2m+1}(z,r_2) + \phi_{-+-}^{(2n+2,2m)}(x^\mu) A_{--}^{2n+2}(y,r_1) A_{++}^{2m}(z,r_2)\Big), \quad (107)$$

$$\phi_{--+}(x^\mu,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big(\phi_{--+}^{(2n+1,2m+1)}(x^\mu) A_{-+}^{2n+1}(y,r_1) A_{-+}^{2m+1}(z,r_2) + \phi_{--+}^{(2n+2,2m+2)}(x^\mu) A_{--}^{2n+2}(y,r_1) A_{--}^{2m+2}(z,r_2)\Big), \quad (108)$$

$$\phi_{---}(x^\mu,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big(\phi_{---}^{(2n+1,2m+2)}(x^\mu) A_{-+}^{2n+1}(y,r_1) A_{--}^{2m+2}(z,r_2) + \phi_{---}^{(2n+2,2m+1)}(x^\mu) A_{--}^{2n+2}(y,r_1) A_{-+}^{2m+1}(z,r_2)\Big). \quad (109)$$

The 4-dimensional fields $\phi^{(n,m)}$ acquire masses $(n^2/r_1^2 + m^2/r_2^2)^{1/2}$ upon the compactification. The zero modes are contained only in the ϕ_{+++} fields. Moreover, because $0 < r_1 \leq R_1$, and $0 < r_2 \leq R_2$, the masses of the KK states $((n^2/r_1^2 + m^2/r_2^2)^{1/2})$ can be set arbitrarily heavy if L_y and L_z are large enough, i.e., we choose suitable (p_i^y, q_i^y) for some i, and (p_j^z, q_j^z) for some j. Therefore, there is no simple relation between the physical size of the extra dimensions and the mass scales of the KK states.

In short, we can have three non-equivalent Z_2 symmetries where two are global symmetries. To be explicit, we would like to give an example: $n = 1$, $2|(p_1^y + q_1^y)$, and $2|(p_1^z + q_1^z)$. In this simple example, the local Z_2 symmetry for the 3-brane can become global if $p_1^y = q_1^y = 1$ and $p_1^z = q_1^z = 1$, and this global symmetry can be moduloed; then, the space-time is $M^4 \times (S^1/Z_2 \times S^1/Z_2)/Z_2$.

6 GUT breaking on the space-time $M^4 \times M^1 \times M^1$

In this section, we would like to discuss the GUT breaking on the space-time $M^4 \times M^1 \times M^1$. As we know, the 6-dimensional $N = 1$ supersymmetric theory is chiral, where the gaugino (and gravitino) has positive chirality and the matter particles (hypermultiplets) have negative chirality, so it often has an anomaly except that we put the standard model fermions on the brane and add a multiplet in the adjoint representation of the gauge group or some other matter contents in the bulk to cancel the gauge anomaly. The 6-dimensional non-supersymmetric GUT models and $N = 1$ supersymmetric GUT models can be considered as special cases of $N = 2$ supersymmetric GUT models, so we only discuss the 6-dimensional $N = 2$ supersymmetric GUT models. The $N = 2$ supersymmetric $SU(5)$, $SU(6)$, $SU(7)$, $SO(10)$, and $SO(12)$ models on the space-time $M^4 \times T^2/(Z_2)^3$ and $M^4 \times T^2/(Z_2)^4$ have been studied completely in [8], and those discussions can be extended to the complete discussions of GUT breaking on the space-time $M^4 \times M^1 \times M^1$ where there are three and four Z_2 symmetries. Because the discussions for GUT breaking are similar, we will not give the complete discussions for the $SU(M)$ and $SO(2M)$ models in this paper. To explain the idea, we will discuss the $N = 2$ supersymmetric $SU(6)$ and $SO(10)$ models on $M^4 \times S^1 \times S^1$ where there are four Z_2 symmetries, and the $N = 2$ supersymmetric $SU(6)$ model with gauge–Higgs unification on $M^4 \times S^1/Z_2 \times S^1/Z_2$ where there are three Z_2 symmetries.

Let us explain the 6-dimensional gauge theory with $N = 2$ supersymmetry. $N = 2$ supersymmetric theory in 6-dimension has 16 real supercharges, corresponding to $N = 4$ supersymmetry in 4-dimension. So, only the vector multiplet can be introduced in the bulk, and we have to put the standard model fermions on the 4-branes, 3-branes or 4-brane intersections. In terms of the 4-dimensional $N = 1$ language, it contains a vector multiplet $V(A_\mu, \lambda_1)$, and three chiral multiplets Σ_5, Σ_6, and Φ. All of these are in the adjoint representation of the gauge group. In addition, the Σ_5 and Σ_6 chiral multiplets contain the gauge fields A_5 and A_6 in their lowest components, respectively.

In the Wess–Zumino gauge and 4-dimensional $N = 1$ language, the bulk action is [14]

$$S = \int d^6x \Bigg\{ \mathrm{Tr}\Bigg[\int d^2\theta \Bigg(\frac{1}{4kg^2}\mathcal{W}^\alpha \mathcal{W}_\alpha + \frac{1}{kg^2}\Big(\Phi\partial_5\Sigma_6 - \Phi\partial_6\Sigma_5 - \frac{1}{\sqrt{2}}\Phi[\Sigma_5,\Sigma_6]\Big)\Bigg) + \mathrm{H.C.}\Bigg] + \int d^4\theta \frac{1}{kg^2}\mathrm{Tr}\Bigg[\sum_{i=5}^{6}\Big((\sqrt{2}\partial_i + \Sigma_i^\dagger)e^{-V}(-\sqrt{2}\partial_i + \Sigma_i)e^V + \partial_i e^{-V}\partial_i e^V\Big) + \Phi^\dagger e^{-V}\Phi e^V\Bigg]\Bigg\}. \quad (110)$$

The gauge transformation is given by

$$\mathrm{e}^V \to \mathrm{e}^{\Lambda}\mathrm{e}^V\mathrm{e}^{\Lambda^\dagger}, \quad (111)$$
$$\Sigma_i \to \mathrm{e}^{\Lambda}(\Sigma_i - \sqrt{2}\partial_i)\mathrm{e}^{-\Lambda}, \quad (112)$$
$$\Phi \to \mathrm{e}^{\Lambda}\Phi\mathrm{e}^{-\Lambda}, \quad (113)$$

where $i = 5, 6$.

From the action, we obtain the vector multiplet transformations under the Z_2 operators P_i^y, P_j^z, $P^{y'z'}$:

$$V(x^\mu, -y_i', z) = P_i^y V(x^\mu, y_i', z)(P^y)^{-1}, \quad (114)$$
$$\Sigma_5(x^\mu, -y_i', z) = -P_i^y \Sigma_5(x^\mu, y_i', z)(P_i^y)^{-1}, \quad (115)$$
$$\Sigma_6(x^\mu, -y_i', z) = P_i^y \Sigma_6(x^\mu, y_i', z)(P_i^y)^{-1}, \quad (116)$$
$$\Phi(x^\mu, -y_i', z) = -P_i^y \Phi(x^\mu, y_i', z)(P_i^y)^{-1}, \quad (117)$$
$$V(x^\mu, y, -z_j') = P_j^z V(x^\mu, y, z_j')(P_j^z)^{-1}, \quad (118)$$
$$\Sigma_5(x^\mu, y, -z_j') = P_j^z \Sigma_5(x^\mu, y, z_j')(P_j^z)^{-1}, \quad (119)$$
$$\Sigma_6(x^\mu, y, -z_j') = -P_j^z \Sigma_6(x^\mu, y, z_j')(P_j^z)^{-1}, \quad (120)$$
$$\Phi(x^\mu, y, -z_j') = -P_j^z \Phi(x^\mu, y, z_j')(P_j^z)^{-1}, \quad (121)$$
$$V(x^\mu, -y_i', -z_i') = P^{y'z'} V(x^\mu, y_i', z_i')(P^{y'z'})^{-1}, \quad (122)$$
$$\Sigma_5(x^\mu, -y_i', -z_i') = -P^{y'z'} \Sigma_5(x^\mu, y_i', z_i')(P^{y'z'})^{-1}, \quad (123)$$
$$\Sigma_6(x^\mu, -y_i', -z_i') = -P^{y'z'} \Sigma_6(x^\mu, y_i', z_i')(P^{y'z'})^{-1}, \quad (124)$$
$$\Phi(x^\mu, -y_i', -z_i') = P^{y'z'} \Phi(x^\mu, y_i', z_i')(P^{y'z'})^{-1}. \quad (125)$$

6.1 $SU(6)$ and $SO(10)$ breaking on $M^4 \times S^1 \times S^1$

In this subsection, we would like to discuss the $SU(6)$ and $SO(10)$ models on $M^4 \times S^1 \times S^1$. We require the following.

(1) There are no zero modes for the chiral multiplets Σ_5, Σ_6 and Φ, and
(2) for the zero modes, we only have the 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^2$ model.

For simplicity, we assume that there are four 4-branes, where two are along the y direction and two along the z direction, $4|(p_1^y + q_1^y)$, and $4|(p_1^z + q_1^z)$. So, we will have four local Z_2 symmetries: P_0^y, P_1^y, P_0^z and P_1^z.

We will choose the unit matrix representations for P_0^y and P_0^z in the adjoint representation of the GUT gauge group. So, considering the zero modes, under a P_0^y projection, we can break the 4-dimensional $N = 4$ supersymmetry to $N = 2$ supersymmetry with (V, Σ_6) forming a vector multiplet and (Σ_5, Φ) forming a hypermultiplet, and we can break the 4-dimensional $N = 2$ supersymmetry to $N = 1$ supersymmetry further by a P_0^z projection.

For a generic bulk field $\phi(x^\mu, y, z)$, we can define four parity operators P_0^y, P_1^y, P_0^z and P_1^z, respectively. Denoting the field with $(P_0^y, P_1^y, P_0^z, P_1^z) = (\pm, \pm, \pm, \pm)$ by $\phi_{\pm\pm\pm\pm}$, we obtain the KK mode expansions, which are those given in (61)–(76).

(I) $SU(6)$ model. We need to choose the matrix representations for the parity operators P_0^y, P_1^y, P_0^z and P_1^z, which are expressed in the adjoint representaion of $SU(6)$. Because $SU(6) \supset SU(5) \times U(1)$, $SU(4) \times SU(2) \times U(1)$, $SU(3) \times SU(3) \times U(1)$, we find that, in general, P_1^y and P_1^z just need to be any two different representations from these three representations: diag$(+1, +1, +1, +1, +1, -1)$, diag$(-1, -1, -1, +1, +1, -1)$, and diag$(-1, -1, -1, +1, +1, +1)$. So the matrix representations for P_0^y, P_0^z, P_1^y and P_1^z are[2]

$$P_0^y = \mathrm{diag}(+1, +1, +1, +1, +1, +1),$$
$$P_0^z = \mathrm{diag}(+1, +1, +1, +1, +1, +1), \quad (126)$$
$$P_1^y = \mathrm{diag}(+1, +1, +1, +1, +1, -1),$$
$$P_1^z = \mathrm{diag}(-1, -1, -1, +1, +1, -1), \quad (127)$$

or

$$P_1^y = \mathrm{diag}(+1, +1, +1, +1, +1, -1),$$
$$P_1^z = \mathrm{diag}(-1, -1, -1, +1, +1, +1), \quad (128)$$

or

$$P_1^y = \mathrm{diag}(-1, -1, -1, +1, +1, +1),$$
$$P_1^z = \mathrm{diag}(-1, -1, -1, +1, +1, -1). \quad (129)$$

Let us point out that

$$SU(6)/\{\mathrm{diag}(+1, +1, +1, +1, +1, -1)\} \approx SU(5) \times U(1), \quad (130)$$
$$SU(6)/\{\mathrm{diag}(-1, -1, -1, +1, +1, -1)\} \approx SU(4) \times SU(2) \times U(1), \quad (131)$$
$$SU(6)/\{\mathrm{diag}(-1, -1, -1, +1, +1, +1)\} \approx SU(3) \times SU(3) \times U(1). \quad (132)$$

(II) $SO(10)$ model. We choose the following matrix representations for the parity operators P_0^y, P_0^z, P_1^y, and P_1^z, which are expressed in the adjoint representaion of $SO(10)$:

$$P_0^y = \mathrm{diag}(+\sigma_0, +\sigma_0, +\sigma_0, +\sigma_0, +\sigma_0), \quad (133)$$
$$P_0^z = \mathrm{diag}(+\sigma_0, +\sigma_0, +\sigma_0, +\sigma_0, +\sigma_0), \quad (134)$$
$$P_1^y = \mathrm{diag}(\sigma_2, \sigma_2, \sigma_2, \sigma_2, \sigma_2), \quad (135)$$
$$P_1^z = \mathrm{diag}(-\sigma_0, -\sigma_0, -\sigma_0, +\sigma_0, +\sigma_0), \quad (136)$$

where σ_0 is the 2×2 unit matrix and σ_2 is the Pauli matrix.

We would like to point out that

$$SO(10)/P_1^y \approx SU(5) \times U(1), \quad (137)$$
$$SO(10)/P_1^z \approx SU(4) \times SU(2) \times SU(2). \quad (138)$$

Now, we discuss the GUT breaking for $SU(6)$ and $SO(10)$ together. Assume $G = SU(6)$ or $G = SO(10)$. Under the P_1^y and P_1^z parities, the gauge generators T^A, where $A = 1, 2, \cdots, 35$ for $SU(6)$ and 45 for $SO(10)$ are separated into four sets: $T^{a,b}$ are the gauge generators for the $SU(3) \times SU(2) \times U(1) \times U(1)$ gauge symmetry, $T^{a,\hat{b}}$,

[2] For the $SU(6)$ model and $SO(10)$ model, one can interchange the matrix representations P_1^y and P_1^z, i.e., $P_1^y \longleftrightarrow P_1^z$, and the discussions are similar

Table 2. Parity assignment and masses ($n \geq 0, m \geq 0$) for the vector multiplet in the $SU(6)$ or $SO(10)$ models on $M^4 \times S^1 \times S^1$

$(P_0^y, P_1^y, P_0^z, P_1^z)$	Field	Mass
$(+,+,+,+)$	$V_\mu^{a,b}$	$\sqrt{(2n)^2/r_1^2 + (2m)^2/r_2^2}$
$(+,+,+,-)$	$V_\mu^{a,\hat{b}}$	$\sqrt{(2n)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(+,-,+,+)$	$V_\mu^{\hat{a},b}$	$\sqrt{(2n+1)^2/r_1^2 + (2m)^2/r_2^2}$
$(+,-,+,-)$	$V_\mu^{\hat{a},\hat{b}}$	$\sqrt{(2n+1)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(-,-,+,+)$	$\Sigma_5^{a,b}$	$\sqrt{(2n+2)^2/r_1^2 + (2m)^2/r_2^2}$
$(-,-,+,-)$	$\Sigma_5^{a,\hat{b}}$	$\sqrt{(2n+2)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(-,+,+,+)$	$\Sigma_5^{\hat{a},b}$	$\sqrt{(2n+1)^2/r_1^2 + (2m)^2/r_2^2}$
$(-,+,+,-)$	$\Sigma_5^{\hat{a},\hat{b}}$	$\sqrt{(2n+1)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(+,+,-,-)$	$\Sigma_6^{a,b}$	$\sqrt{(2n)^2/r_1^2 + (2m+2)^2/r_2^2}$
$(+,+,-,+)$	$\Sigma_6^{a,\hat{b}}$	$\sqrt{(2n)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(+,-,-,-)$	$\Sigma_6^{\hat{a},b}$	$\sqrt{(2n+1)^2/r_1^2 + (2m+2)^2/r_2^2}$
$(+,-,-,+)$	$\Sigma_6^{\hat{a},\hat{b}}$	$\sqrt{(2n+1)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(-,-,-,-)$	$\Phi^{a,b}$	$\sqrt{(2n+2)^2/r_1^2 + (2m+2)^2/r_2^2}$
$(-,-,-,+)$	$\Phi^{a,\hat{b}}$	$\sqrt{(2n+2)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(-,+,-,-)$	$\Phi^{\hat{a},b}$	$\sqrt{(2n+1)^2/r_1^2 + (2m+2)^2/r_2^2}$
$(-,+,-,+)$	$\Phi^{\hat{a},\hat{b}}$	$\sqrt{(2n+1)^2/r_1^2 + (2m+1)^2/r_2^2}$

$T^{\hat{a},b}$, and $T^{\hat{a},\hat{b}}$ are the other broken gauge generators which belong to $\{G/P_1^y \cap \{\text{coset}G/P_1^z\}\}$, $\{\{\text{coset}G/P_1^y\} \cap G/P_1^z\}$, and $\{\{\text{coset}G/P_1^y\} \cap \{\text{coset}G/P_1^z\}\}$, respectively. Therefore, under P_0^y, P_0^z, P_1^y and P_1^z, the gauge generators transform as

$$P_0^y T^{A,B}(P_0^y)^{-1} = T^{A,B}, \quad P_0^z T^{A,B}(P_0^z)^{-1} = T^{A,B}, \quad (139)$$

$$P_1^y T^{a,B}(P_1^y)^{-1} = T^{a,B}, \quad P_1^y T^{\hat{a},B}(P_1^y)^{-1} = -T^{\hat{a},B}, \quad (140)$$

$$P_1^z T^{A,b}(P_1^z)^{-1} = T^{A,b}, \quad P_1^z T^{A,\hat{b}}(P_1^z)^{-1} = -T^{A,\hat{b}}. \quad (141)$$

The particle spectra are given in Table 2, and the gauge superfields, the number of 4-dimensional supersymmetry and the gauge group on the 4-branes or 4-brane intersections are given in Table 3. For the zero modes, we only have 4-dimensional $N=1$ supersymmetric $SU(3) \times SU(2) \times U(1) \times U(1)$ model in the bulk. From Table 3, we see that, including the KK modes, the 3-brane (the intersection of 4-branes) and 4-brane preserve $N=1$ and $N=2$ supersymmetry, respectively. The gauge group on the 3-brane can be G, or G/P_1^z, or G/P_1^y, or $SU(3) \times SU(2) \times U(1) \times U(1)$. And the gauge group on the 4-brane can be G, or G/P_1^y or G/P_1^z. The phenomenology discussions are similar to those in [8].

6.2 $SU(6)$ breaking on $M^4 \times S^1/Z_2 \times S^1/Z_2$

In this subsection, we would like to discuss the $SU(6)$ model on $M^4 \times S^1/Z_2 \times S^1/Z_2$ in which there are two global Z_2 symmetries and one local Z_2 symmetry. Although we cannot project out all the zero modes for the chiral multiplets Σ_5, Σ_6 and Φ, we require that the extra zero modes are only from Φ and form two Higgs doublets, which is called gauge–Higgs unification. Our basic requirements are
(1) there are no zero modes for the chiral multiplets Σ_5 and Σ_6;
(2) considering the zero modes, there is only one pair of Higgs doublets because if we had two pairs of Higgs doublets, we may have a flavour changing neutral current problem.

For simplicity, we assume that there are only four 4-branes, which are the boundaries for $S^1/Z_2 \times S^1/Z_2$, there is only one 3-brane in the bulk, and $2|(p_1^y + q_1^y)$, $2|(p_1^z + q_1^z)$. So we will have two global Z_2 symmetries P^y and P^z, and one local Z_2 symmetry $P^{y'z'}$. In order to project out all the zero modes of Σ_5 and Σ_6, we would like to choose the matrix representation of P^y equal to that of P^z.

For a generic bulk field $\phi(x^\mu, y, z)$, we can define three parity operators P^y, P^z, $P^{y'z'}$, respectively. Denoting the field by $(P^y, P^z, P^{y'z'}) = (\pm, \pm, \pm)$ by $\phi_{\pm\pm\pm}$, we obtain the KK mode expansions which are given in (102)–(109).

There are two scenarios which have gauge–Higgs unification.
(1) We choose the matrix representations for P^y, P^z and $P^{y'z'}$ as follows:

$$P^y = P^z = \text{diag}(+1,+1,+1,+1,+1,-1), \quad (142)$$

$$P^{y'z'} = \text{diag}(-1,-1,-1,+1,+1,+1). \quad (143)$$

(2) We choose the matrix representations for P^y, P^z, and $P^{y'z'}$ as follows:

$$P^y = P^z = \text{diag}(-1,-1,-1,+1,+1,-1), \quad (144)$$

$$P^{y'z'} = \text{diag}(-1,-1,-1,+1,+1,+1). \quad (145)$$

Under P^y (or P^z) and $P^{y'z'}$ parities, the gauge generators T^A, where $A = 1, 2, \cdots, 35$ for $SU(6)$ are separated into four sets: $T^{a,b}$ are the gauge generators for $SU(3) \times SU(2) \times U(1) \times U(1)$ gauge symmetry, $T^{a,\hat{b}}$, $T^{\hat{a},b}$, and $T^{\hat{a},\hat{b}}$ are the other broken gauge generators which belong to $\{G/P^y \cap \{\text{coset}G/P^{y'z'}\}\}$, $\{\{\text{coset}G/P^y\} \cap G/P^{y'z'}\}$, and $\{\{\text{coset}G/P^y\} \cap \{\text{coset}G/P^{y'z'}\}\}$, respectively. Therefore, under P^y, P^z, and $P^{y'z'}$, the gauge generators transform as

$$P^y T^{a,B}(P^y)^{-1} = T^{a,B},$$
$$P^y T^{\hat{a},B}(P^y)^{-1} = -T^{\hat{a},B}, \quad (146)$$

$$P^z T^{a,B}(P^z)^{-1} = T^{a,B},$$
$$P^z T^{\hat{a},B}(P^z)^{-1} = -T^{\hat{a},B}, \quad (147)$$

$$P^{y'z'} T^{A,b}(P^{y'z'})^{-1} = T^{A,b},$$
$$P^{y'z'} T^{A,\hat{b}}(P^{y'z'})^{-1} = -T^{A,\hat{b}}. \quad (148)$$

We present the particle spectra in Table 4, and the number of 4-dimensional supersymmetry and gauge symmetries on the 3-brane, 4-brane intersections and 4-branes in Table 5. Because the 3-brane is the fixed point under

Table 3. For the model $G = SU(6)$ or $G = SO(10)$ on $M^4 \times S^1 \times S^1$, the gauge superfields, the number of 4-dimensional supersymmetry and gauge symmetries on the interesections of 4-branes, which are located at $(y = 0, z = 0)$, $(y = 0, z = z_1)$, $(y = y_1, z = 0)$, and $(y = y_1, z = z_1)$, or on the 4-branes which are located at $y = 0$, $z = 0$, $y = y_1$, $z = z_1$

Brane position	Fields	SUSY	Gauge symmetry
$(0, 0)$	$V_\mu^{A,B}$	$N = 1$	G
$(0, z_1)$	$V_\mu^{A,b}$, $\Sigma_6^{A,\hat{b}}$	$N = 1$	G/P_1^z
$(y_1, 0)$	$V_\mu^{a,B}$, $\Sigma_5^{\hat{a},B}$	$N = 1$	G/P_1^y
(y_1, z_1)	$V_\mu^{a,b}$, $\Sigma_5^{\hat{a},b}$, $\Sigma_6^{a,\hat{b}}$, $\Phi^{\hat{a},\hat{b}}$	$N = 1$	$SU(3) \times SU(2) \times U(1) \times U(1)$
$y = 0$	$V_\mu^{A,B}$, $\Sigma_6^{A,B}$	$N = 2$	G
$z = 0$	$V_\mu^{A,B}$, $\Sigma_5^{A,B}$	$N = 2$	G
$y = y_1$	$V_\mu^{a,B}$, $\Sigma_5^{\hat{a},B}$, $\Sigma_6^{a,B}$, $\Phi^{\hat{a},B}$	$N = 2$	G/P_1^y
$z = z_1$	$V_\mu^{A,b}$, $\Sigma_5^{A,b}$, $\Sigma_6^{A,\hat{b}}$, $\Phi^{A,\hat{b}}$	$N = 2$	G/P_1^z

$P^{y'z'}$ symmetry, it preserves 4-dimensional $N = 2$ supersymmetry. The 4-branes are the fixed lines under one global symmetry, the 4-brane intersections and 4-branes preserve 4-dimensional $N = 1$ and $N = 2$ supersymmetry, respectively. The phenomenology discussions are also similar to those in [8].

7 Discrete symmetry on the space-time $M^4 \times A^2$

As we know, in each point of the 2-dimensional real manifold, there is an open neighborhood homeomorphic to R^2 in real coordinates or C^1 in complex coordinates, and its rotation group is locally $SO(2)$ or $U(1)$. So we may define the Z_n discrete symmetry on the 2-dimensional manifold in which n is any positive integer and we can break any supersymmetric $SU(M)$ GUT models down to the 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{M-4}$ models for the zero modes. One obvious candidate for the extra space manifold is the disc D^2 where there is one 3-brane at the origin and one 4-brane at the outer boundary. To be general, we can consider that the extra space manifold is the annulus where there are two boundary 4-branes. For simplicity, we denote the annulus by A^2.

Furthermore, we discuss the KK mode expansions on the space-time $M^4\times$ (a segment of A^2), and find that the masses of the KK states might be set arbitrarily heavy if the range of the angle is small enough.

7.1 Discrete symmetry on $M^4\times$ A^2

We consider that the extra space manifold is the annulus A^2. Convenient coordinates for the annulus A^2 are the polar coordinates (r, θ), and it is easy to change these to complex coordinates by $z = r\mathrm{e}^{\mathrm{i}\theta}$. We assume that the innner radius of the annulus is R_1, and the outer radius of the annulus is R_2. Taking $R_1 = 0$, we obtain the disc D^2. Considering Z_n symmetry on A^2, we define

Table 4. Parity assignment and masses ($n \geq 0, m \geq 0$) for the vector multiplet in the $SU(6)$ model on $M^4 \times S^1/Z_2 \times S^1/Z_2$

$(P^y, P^z, P^{y'z'})$	Field	Mass
$(+, +, +)$	V_μ^{ab}, $\Phi^{\hat{a}\hat{b}}$	$\sqrt{(2n)^2/r_1^2 + (2m)^2/r_2^2}$ or $\sqrt{(2n+1)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(+, +, -)$	$V_\mu^{a\hat{b}}$, $\Phi^{\hat{a}\hat{b}}$	$\sqrt{(2n+1)^2/r_1^2 + (2m)^2/r_2^2}$ or $\sqrt{(2n)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(+, -, +)$	$\Sigma_5^{\hat{a}\hat{b}}$, $\Sigma_6^{a\hat{b}}$	$\sqrt{(2n)^2/r_1^2 + (2m+1)^2/r_2^2}$ or $\sqrt{(2n+1)^2/r_1^2 + (2m+2)^2/r_2^2}$
$(+, -, -)$	$\Sigma_5^{\hat{a}b}$, Σ_6^{ab}	$\sqrt{(2n)^2/r_1^2 + (2m+2)^2/r_2^2}$ or $\sqrt{(2n+1)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(-, +, +)$	$\Sigma_5^{a\hat{b}}$, $\Sigma_6^{\hat{a}\hat{b}}$	$\sqrt{(2n+1)^2/r_1^2 + (2m)^2/r_2^2}$ or $\sqrt{(2n+2)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(-, +, -)$	Σ_5^{ab}, $\Sigma_6^{\hat{a}b}$	$\sqrt{(2n+2)^2/r_1^2 + (2m)^2/r_2^2}$ or $\sqrt{(2n+1)^2/r_1^2 + (2m+1)^2/r_2^2}$
$(-, -, +)$	$V_\mu^{\hat{a}b}$, Φ^{ab}	$\sqrt{(2n+1)^2/r_1^2 + (2m+1)^2/r_2^2}$ or $\sqrt{(2n+2)^2/r_1^2 + (2m+2)^2/r_2^2}$
$(-, -, -)$	$V_\mu^{\hat{a}\hat{b}}$, $\Phi^{a\hat{b}}$	$\sqrt{(2n+2)^2/r_1^2 + (2m+1)^2/r_2^2}$ or $\sqrt{(2n+1)^2/r_1^2 + (2m+2)^2/r_2^2}$

$$\omega = \mathrm{e}^{\mathrm{i}2\pi/n}, \tag{149}$$

and we define the generator for Z_n as Ω which satifies $\Omega^n = 1$.

For a generic bulk multiplet Φ which fills a representation of the bulk gauge group G, we have

$$\begin{aligned}\Omega\Phi(x^\mu, z, \bar{z}) &= \Phi(x^\mu, \omega z, \omega^{n-1}\bar{z}) \\ &= \eta_\Phi (R_\Omega)^{l_\Phi} \Phi(x^\mu, z, \bar{z})(R_\Omega^{-1})^{m_\Phi},\end{aligned} \tag{150}$$

where $\eta_\Phi \subset Z_n$, and R_Ω is an element in the adjoint representation of G which satisfies $R_\Omega^n = 1$.

Table 5. The $G = SU(6)$ model with gauge–Higgs unification on $M^4 \times S^1/Z_2 \times S^1/Z_2$. The gauge superfield V_μ, the number of 4-dimensional supersymmetry and gauge symmetries on the 4-brane intersections or 3-brane, which are located at the $(y = 0, z = 0)$, $(y = 0, z = \pi R_2)$, $(y = \pi R_1, z = 0)$, $(y = \pi R_1, z = \pi R_2)$, and $(y = y_1, z = z_1)$, and on the 4-branes which are located at the fixed lines $y = 0$, $y = \pi R_1$, $z = 0$, $z = \pi R_2$

Brane position	Fields	SUSY	Gauge symmetry
$(0,0)$, $(0,\pi R_2)$, $(\pi R_1, 0)$, $(\pi R_1, \pi R_2)$,	$V_\mu^{a,B}$	$N = 1$	G/P^y or G/P^z
(y_1, z_1)	$V_\mu^{A,b}$	$N = 2$	$G/P^{y'z'}$
$y = 0$, $y = \pi R_1$, $z = 0$, $z = \pi R_2$	$V_\mu^{a,B}$	$N = 2$	G/P^y or G/P^z

A generic field $\phi(x^\mu, z, \bar{z})$ with eigenvalue ω^l under the operator Ω we write as $\phi_{\omega^l}(x^\mu, z, \bar{z})$, i.e.,

$$\Omega \phi_{\omega^l}(x^\mu, z, \bar{z}) = \omega^l \phi_{\omega^l}(x^\mu, z, \bar{z}). \tag{151}$$

The KK modes expansions for $\phi_{\omega^l}(x^\mu, z, \bar{z})$ are

$$\phi_{\omega^l}(x^\mu, z, \bar{z}) = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} \phi_{\omega^l}^{(jk)}(x^\mu) f_{jk}^{\omega^l}(z, \bar{z}), \tag{152}$$

where $l = 0, 1, \cdots, n-1$, and

$$f_{jk}^{\omega^l}(z, \bar{z}) = \sum_{s=0}^{n-1} \omega^{(n-s)l} f_{jk}(\omega^s z, \omega^{n-s} \bar{z}). \tag{153}$$

The functions $f_{jk}(z, \bar{z})$ are defined by

$$f_{jk}(z, \bar{z}) = J_j(\lambda_{jk} r) \mathrm{e}^{\mathrm{i} j \theta}, \tag{154}$$

or

$$f_{jk}(z, \bar{z}) = J_j(\lambda_{jk} |z|)(z/|\bar{z}|)^j, \tag{155}$$

where $J_j(\lambda_{jk} r)$ is the first order Bessel function, satisfying the Dirichlet or Neumann boundary condition at $r = R_1$ and $r = R_2$:

$$J_j(\lambda_{jk} r) = 0 \text{ or } \frac{\mathrm{d} J_j(\lambda_{jk} r)}{\mathrm{d} \lambda_{jk} r} = 0, \quad \text{for } r = R_1 \text{ and } R_2. \tag{156}$$

The zero modes are contained only in the ϕ_{ω^0} fields, i.e. $l = 0$.

First, we consider that the extra space manifold is a disc D^2, i.e., $R_1 = 0$, so we only have the boundary condition at $r = R_2$, and then we can have all the KK states. There is also one fixed point under the Z_n symmetry, which is the center of the disc. One might wonder whether there is a singularity for $f_{jk}(z, \bar{z})$ at the origin $r = 0$ when $j \neq 0$; however, there is no singularity and we only have zero modes at the origin because $J_0(0) = 1$ and $J_s(0) = 0$ for $s \geq 1$. By the way, the global Z_n symmetry can be moduloed from D^2, and the corresponding orbifold is D^2/Z_n.

Second, we consider that the extra space manifold is an annulus A^2. The Z_n symmetry acts free on the annulus A^2, so we can obtain the quotient manifold A^2/Z_n by moduloing the Z_n symmetry. We will also have much less KK states, i.e., a lot of KK states in the summation might be absent because the boundary conditions at $r = R_1$ and $r = R_2$ must be satisfied simultaneously. The interesting phenomenology is that we might have the scenario in which only a few KK states are light and the other KK states are relatively heavy, so we may produce the light KK states of the gauge fields at future colliders.

7.2 KK modes on $M^4 \times$ (a segment of A^2)

If the extra space manifold is a segment of A^2, we would like to discuss the KK modes. We assume that the inner radius of the annulus is R_1, the outer radius of the annulus is R_2, and the angle θ is

$$0 \leq \theta \leq \alpha 2\pi, \tag{157}$$

where $0 < \alpha < 1$. The KK modes expansion for a generic field ϕ is

$$\phi(x^\mu, z, \bar{z}) = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} \phi^{(jk)}(x^\mu) f_{jk}(z, \bar{z}), \tag{158}$$

where

$$f_{jk}(z, \bar{z}) = J_{j/(4\alpha)}(\lambda_{jk} r) \mathrm{e}^{\mathrm{i} j \theta/(4\alpha)}, \tag{159}$$

or

$$f_{jk}(z, \bar{z}) = J_{j/(4\alpha)}(\lambda_{jk} |z|)(z/|\bar{z}|)^{j/(4\alpha)}. \tag{160}$$

At $r = R_1$ and $r = R_2$, the function $J_{j/(4\alpha)}(\lambda_{jk} r)$ should satisfy the Dirichlet boundary condition or Neumann condition,

$$J_j(\lambda_{jk} r) = 0 \text{ or } \frac{\mathrm{d} J_j(\lambda_{jk} r)}{\mathrm{d} \lambda_{jk} r} = 0. \tag{161}$$

In short, if α is very small, then $j/(4\alpha)$ will be very large, and the KK states may be set arbitrarily heavy, which is similar to [11].

We can define the Z_2 reflection symmetry on the sector of D^2 or the segment of A^2. However, we cannot define the discrete symmetry Z_n for $n > 2$ on the sector of D^2 or segment of A^2, so it is not interesting for us to discuss the supersymmetric GUT breaking in this case.

8 GUT breaking on the space-time $M^4 \times A^2$

In this section, we would like to discuss the 6-dimensional $N = 2$ supersymmetric $SU(M)$ GUT models on the space-time $M^4 \times A^2$ or $M^4 \times D^2$.

In principle, we can break any $SU(M)$ gauge group on the space-time $M^4 \times A^2$ or $M^4 \times D^2$, because we can choose a Z_n symmetry in which n is very large. As an example, we will discuss the $SU(6)$ models on the space-time $M^4 \times A^2$ or $M^4 \times D^2$ with Z_9 symmetry, or one can consider the space-time $M^4 \times A^2/Z_9$ or $M^4 \times D^2/Z_9$.

The $N = 2$ supersymmetry in 6-dimension corresponds to $N = 4$ supersymmetry in 4-dimension; thus, only the gauge multiplet can be introduced in the bulk. This multiplet can be decomposed under the 4-dimensional $N = 1$ supersymmetry into a vector multiplet V and three chiral multiplets Σ, Φ, and Φ^{c} in the adjoint representation, with the fifth and sixth components of the gauge field, A_5 and A_6, contained in the lowest component of Σ. The standard model fermions are on the boundary 4-brane at $r = R_1$ or $r = R_2$ for the annulus A^2, and on the 3-brane at the origin or on the boundary 4-brane at $r = R_2$ for the disc D^2.

In the Wess–Zumino gauge and 4-dimensional $N = 1$ language, the bulk action is [14]

$$
\begin{aligned}
S = \int \mathrm{d}^6 x \Bigg\{ \mathrm{Tr} \Bigg[\int \mathrm{d}^2\theta \left(\frac{1}{4kg^2} \mathcal{W}^\alpha \mathcal{W}_\alpha \right. & \qquad (162) \\
+ \left. \frac{1}{kg^2} \left(\Phi^{\mathrm{c}} \partial \Phi - \frac{1}{\sqrt{2}} \Sigma [\Phi, \Phi^{\mathrm{c}}] \right) \right) + \mathrm{h.c.} \Bigg] & \\
+ \int \mathrm{d}^4\theta \frac{1}{kg^2} \mathrm{Tr} \left[(\sqrt{2}\partial^\dagger + \Sigma^\dagger) \mathrm{e}^{-V} (-\sqrt{2}\partial + \Sigma) \mathrm{e}^V \right] & \\
+ \int \mathrm{d}^4\theta \frac{1}{kg^2} \mathrm{Tr} \left[+ \Phi^\dagger \mathrm{e}^{-V} \Phi \mathrm{e}^V + \Phi^{\mathrm{c}\dagger} \mathrm{e}^{-V} \Phi^{\mathrm{c}} \mathrm{e}^V \right] \Bigg\}. &
\end{aligned}
$$

Notice that if we consider Z_9 symmetry, then $\omega = \mathrm{e}^{\mathrm{i}2\pi/9}$. From the above action, we obtain the transformations of the gauge multiplet under Ω:

$$V(\omega z, \omega^8 \bar{z}) = R_\Omega V(z, \bar{z}) R_\Omega^{-1}, \quad (163)$$

$$\Sigma(\omega z, \omega^8 \bar{z}) = \omega^8 R_\Omega \Sigma(z, \bar{z}) R_\Omega^{-1}, \quad (164)$$

$$\Phi(\omega z, \omega^8 \bar{z}) = \omega^m R_\Omega \Phi(z, \bar{z}) R_\Omega^{-1}, \quad (165)$$

$$\Phi^{\mathrm{c}}(\omega z, \omega^8 \bar{z}) = \omega^{10-m} R_\Omega \Phi^{\mathrm{c}}(z, \bar{z}) R_\Omega^{-1}, \quad (166)$$

where $0 \le m \le 8$; R_Ω is an element in the adjoint representation of the GUT gauge group and satisfies the equation $R_\Omega^9 = 1$. To be compatible with our previous discussions in Sect. 6, we choose $m = 8$, and then

$$\Phi(\omega z, \omega^8 \bar{z}) = \omega^8 R_\Omega \Phi(z, \bar{z}) R_\Omega^{-1}, \quad (167)$$

$$\Phi^{\mathrm{c}}(\omega z, \omega^8 \bar{z}) = \omega^2 R_\Omega \Phi^{\mathrm{c}}(z, \bar{z}) R_\Omega^{-1}. \quad (168)$$

Now, we would like to discuss the supersymmetric $SU(6)$ model. We choose the following matrix representations for the Z_9 operator Ω, R_Ω, which are expressed in the adjoint representation of $SU(6)$:

$$R_\Omega = \mathrm{diag}(\omega^2, \omega^2, \omega^2, \omega^8, \omega^8, \omega^5). \quad (169)$$

So, upon using the Z_9 operator Ω, the gauge generators T^A, where $A = 1, 2, \cdots, 35$ for $SU(6)$ are separated into two sets: T^a are the gauge generators for the $SU(3) \times SU(2) \times U(1)^2$ gauge group, and $T^{\hat{a}}$ are the other broken gauge generators:

$$R_\Omega T^a R_\Omega^{-1} = T^a, \quad R_\Omega T^{\hat{a}} \, R_\Omega^{-1} = -T^{\hat{a}}. \quad (170)$$

First, we consider that the extra space manifold is the annulus A^2. For the zero modes, we have 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^2$ gauge symmetry in the bulk and on the 4-branes at $r = R_1$ and $r = R_2$. Including the KK states, we will have the 4-dimensional $N = 4$ supersymmetry and $SU(6)$ gauge symmetry in the bulk, and on the 4-branes at $r = R_1$ and $r = R_2$.

Second, we consider that the extra space manifold is the disc D^2. For the zero modes, we have 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^2$ gauge symmetry in the bulk and on the 4-brane at $r = R_2$. Including all the KK states, we will have the 4-dimensional $N = 4$ supersymmetry and $SU(6)$ gauge symmetry in the bulk, and on the 4-brane at $r = R_2$. In addition, we always have 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^2$ gauge symmetry on the 3-brane at the origin in which only the zero modes exist. So, if we put the standard model fermions on the 3-brane at the origin, the extra dimensions can be large and the gauge hierarchy problem can be solved, for there does not exist a proton decay problem at all.

In order to break the extra $U(1)$ symmetry, we have to introduce the extra chiral multiplets which are singlets under the standard model gauge symmetry, and use the Higgs mechanism. If we considered the chiral model on the observable brane, we will have to introduce the exotic particles due to the anomaly cancellation.

In short, we can introduce Z_n symmetry to break any supersymmetric $SU(M)$ GUT models as long as n is large enough. There are 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^{M-4}$ gauge symmetries in the bulk and on the 4-branes for the zero modes, and on the 3-brane at the origin in the disc D^2 scenario. Including all the KK states, we will have the 4-dimensional $N = 4$ supersymmetry and $SU(M)$ gauge symmetry in the bulk, and on the 4-branes. In addition, the standard model fermions are on the boundary 4-brane at $r = R_1$ or $r = R_2$ if the extra space manifold is the annulus A^2, and on the 3-brane at the origin or on the boundary 4-brane at $r = R_2$ if the extra space manifold is the disc D^2.

9 Discrete symmetry on the space-time $M^4 \times T^2$

In this section, we would like to discuss the discrete symmetry on the space-time $M^4 \times T^2$. Because for any point in T^2, there is an open neighborhood which is homeomorphic to R^2, naively, one might think that one can introduce any Z_n symmetry on T^2. However, this is not true. In fact, we can prove that the only discrete symmetries on the torus from the rotation group $SO(2)$ or $U(1)$ are Z_2, Z_3, Z_4, and Z_6.

The proof is as follows. In complex coordinates, the torus T^2 can be defined by C^1 modulo the equivalent classes: $z \sim z + 2\pi R_1$ and $z \sim z + 2\pi R_2\omega$ where $\omega = \mathrm{e}^{\mathrm{i}\theta}$. So we need to discuss the Z_2, Z_3 and Z_n symmetries for $n > 3$ separately.

First, we consider Z_2 symmetry; the equivalent class is $z \sim -z$, and the two fixed points are $z = 0$ and $z = \pi R_1 + \pi R_2 \omega$. Therefore, the global Z_2 symmetry can be defined on the general torus T^2. The 3-branes can be located at the fixed points.

Second, we consider Z_3 symmetry. We have to choose $R_1 = R_2 = R$. Define $\theta = 2\pi/3$; we obtain the equivalent class $z \sim z + 2\pi R\omega$ equivalent to the equivalence class $z \sim z + 2\pi R \mathrm{e}^{\mathrm{i}\pi/3}$. There are three fixed points: $z = 0$, $z = 2\pi R \mathrm{e}^{\mathrm{i}\pi/6}/3^{1/2}$, and $z = 4\pi R \mathrm{e}^{\mathrm{i}\pi/6}/3^{1/2}$. The 3-branes can be located at the fixed points.

Third, in order to define the Z_n symmetry for $n > 3$, we have to choose $\theta = 2\pi/n$ and $R_1 = R_2 = R$. In addition, ω should satisfy the following equations:

$$2\pi R\omega = l' 2\pi R + k' 2\pi R\omega, \tag{171}$$

$$2\pi R\omega\omega = l 2\pi R + k 2\pi R\omega, \tag{172}$$

where l, k, l' and k' are integers. The first equation is satisfied by choosing $l' = 0$ and $k' = 1$. Moreover, from the second equation, we obtain

$$\omega = \frac{l \pm \sqrt{l^2 + 4k}}{2}. \tag{173}$$

The complete solutions are $l = 0$ and $k = \pm 1$, $l = 1$ and $k = -1$. Then all the possible ω are ± 1, $\pm \mathrm{i}$, $\mathrm{e}^{\mathrm{i}2\pi/3}$, $\mathrm{e}^{\mathrm{i}2\pi/6}$. Therefore, the discrete symmetries Z_n for $n > 3$ on the torus from the rotation group $SO(2)$ or $U(1)$ are Z_4 and Z_6. Moreover, for the Z_4 discrete symmetry, there are two Z_4 fixed points: $z = 0$ and $z = 2^{1/2}\pi R \mathrm{e}^{\mathrm{i}\pi/4}$, two Z_2 fixed points: $z = \pi R$ and $z = \pi R \mathrm{e}^{\mathrm{i}\pi/2}$. For Z_6 discrete symmetry, there is one Z_6 fixed point $z = 0$, and there are two Z_3 fixed points: $z = 2\pi R \mathrm{e}^{\mathrm{i}\pi/6}/3^{1/2}$ and $z = 4\pi R \mathrm{e}^{\mathrm{i}\pi/6}/3^{1/2}$, and three Z_2 fixed points: $z = 3^{1/2}\pi R \mathrm{e}^{\mathrm{i}\pi/6}$, $z = \pi R$ and $z = \pi R \mathrm{e}^{\mathrm{i}\pi/3}$. The 3-branes can be located at those fixed points.

For the general T^2 defined by the equivalence classes $z \sim z + 2\pi R_1$ and $z \sim z + 2\pi R_2 \mathrm{e}^{\mathrm{i}\theta}$, the KK modes expansion for a generic field ϕ is

$$\phi(x^\mu, z, \bar{z}) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \phi^{(jk)}(x^\mu) f_{jk}(z, \bar{z}), \tag{174}$$

where

$$f_{jk}(z, \bar{z}) = \exp\{\mathrm{i}[(a - \mathrm{i}b)z + (a + \mathrm{i}b)\bar{z}]\}, \tag{175}$$

and

$$a = \frac{j}{2R_1}, \tag{176}$$

$$b = \frac{1}{\sin\theta}\left(\frac{k}{2R_2} - \frac{j}{2R_1}\cos\theta\right). \tag{177}$$

The mass for ϕ^{jk} is

$$M^2_{\phi^{jk}} = \frac{1}{\sin\theta^2}\left[\frac{j^2}{4R_1^2} + \frac{k^2}{4R_2^2} - \frac{2jk}{4R_1R_2}\cos\theta\right]. \tag{178}$$

The zero modes are contained in $\phi^{(00)}$ sector, i.e., $j = 0$ and $k = 0$.

The fundamental group for the torus is $Z \bigoplus Z$, so one might think we can break the gauge symmetry by a Wilson line in the mean time. The key question is how to define the suitable KK mode expansions for the bulk fields in the Wilson line approach, because we require that the fields without zero modes should have KK mode excitations.

10 GUT breaking on the space-time $M^4 \times T^2$

We would like to consider the 6-dimensional $N = 2$ supersymmetric $SU(5)$ model on $M^4 \times T^2$ with Z_6 symmetry.

Notice that we consider Z_6 symmetry; we define $\omega = \mathrm{e}^{\mathrm{i}2\pi/6}$. From the action in (162), we obtain the transformations of the gauge multiplet under Ω:

$$V(\omega z, \omega^5 \bar{z}) = R_\Omega V(z, \bar{z}) R_\Omega^{-1}, \tag{179}$$

$$\Sigma(\omega z, \omega^5 \bar{z}) = \omega^5 R_\Omega \Sigma(z, \bar{z}) R_\Omega^{-1}, \tag{180}$$

$$\Phi(\omega z, \omega^5 \bar{z}) = \omega^5 R_\Omega \Phi(z, \bar{z}) R_\Omega^{-1}, \tag{181}$$

$$\Phi^{\mathrm{c}}(\omega z, \omega^5 \bar{z}) = \omega^2 R_\Omega \Phi^{\mathrm{c}}(z, \bar{z}) R_\Omega^{-1}, \tag{182}$$

where R_Ω is an element in the adjoint representation of $SU(5)$ and satisfies the equation $R_\Omega^6 = 1$.

We choose the following matrix representations for the Z_6 operator Ω, R_Ω, which are expressed in the adjoint representation of $SU(5)$:

$$R_\Omega = \mathrm{diag}(1, 1, 1, -1, -1). \tag{183}$$

So, upon using the Z_6 operator Ω, the gauge generators T^A, where $A = 1, 2, \cdots, 24$, for $SU(5)$ are separated into two sets: T^a are the gauge generators for the standard model gauge group, and $T^{\hat{a}}$ are the other broken gauge generators

$$R_\Omega T^a R_\Omega^{-1} = T^a, \quad R_\Omega T^{\hat{a}} R_\Omega^{-1} = -T^{\hat{a}}. \tag{184}$$

In addition, the representation of the generator for Z_2 symmetry is

$$R_\Omega^3 = \mathrm{diag}(1, 1, 1, -1, -1), \tag{185}$$

and the representation of the generator for Z_3 symmetry is

$$R_\Omega^2 = \mathrm{diag}(1, 1, 1, 1, 1). \tag{186}$$

Therefore, the gauge symmetries on the 3-branes at the Z_2 fixed points and on the 3-branes at the Z_3 fixed points are

$$SU(5)/R_\Omega^3 \approx SU(3) \times SU(2) \times U(1), \tag{187}$$

$$SU(5)/R_\Omega^2 \approx SU(5). \tag{188}$$

Table 6. The $G = SU(5)$ model on $M^4 \times T^2$ with Z_6 symmetry. The gauge multiplet, the number of 4-dimensional supersymmetry and gauge symmetry on the 3-branes, which are located at the Z_6 fixed point $z = 0$; the Z_3 fixed points: $z = 2\pi R \mathrm{e}^{\mathrm{i}\pi/6}/3^{1/2}$, and $z = 4\pi R \mathrm{e}^{\mathrm{i}\pi/6}/3^{1/2}$; and the Z_2 fixed points: $z = 3^{1/2}\pi R \mathrm{e}^{\mathrm{i}\pi/6}$, $z = \pi R$, and $z = \pi R \mathrm{e}^{\mathrm{i}\pi/3}$

Brane position	Fields	SUSY	Gauge symmetry
$z = 0$	V^a	$N = 1$	$SU(3) \times SU(2) \times U(1)$
$z = 2\pi R \mathrm{e}^{\mathrm{i}\pi/6}/3^{1/2}$, $z = 4\pi R \mathrm{e}^{\mathrm{i}\pi/6}/3^{1/2}$	V^A, Σ^a, Φ^a, $(\Phi^c)^{\hat{a}}$	$N = 1$	$SU(5)$
$z = 3^{1/2}\pi R \mathrm{e}^{\mathrm{i}\pi/6}$, $z = \pi R$, $z = \pi R \mathrm{e}^{\mathrm{i}\pi/3}$	V^a, Σ^A, Φ^A, $(\Phi^c)^A$	$N = 4$	$SU(3) \times SU(2) \times U(1)$

The gauge multiplet, the number of 4-dimensional supersymmetry and gauge symmetries on the 3-branes are given in Table 6. In short, we have 4-dimensional $N = 1$ supersymmetry and the standard model gauge symmetry in the bulk for the zero modes, and on the 3-brane at Z_6 fixed point for all the modes. Including the KK states, we will have the 4-dimensional $N = 4$ supersymmetry and $SU(5)$ gauge symmetry in the bulk, the 4-dimensional $N = 1$ supersymmetry and $SU(5)$ gauge symmetry on the 3-branes at Z_3 fixed points, and the 4-dimensional $N = 4$ supersymmetry and $SU(3) \times SU(2) \times U(1)$ gauge symmetry on the 3-branes at Z_2 fixed points. The standard model fermions and Higgs fields can be on any 3-brane at one of the fixed points. In particular, if we put the standard model fermions and Higgs fields on the 3-brane at the Z_6 fixed point, the extra dimensions can be large and the gauge hierarchy problem can be solved because there is no proton decay problem at all.

11 Discussion and conclusion

With the ansatz that there exist local or global discrete symmetries in the special branes' neighborhoods, we discuss the general reflection Z_2 symmetries on the space-time $M^4 \times M^1$ and $M^4 \times M^1 \times M^1$. We find that we can have at most two Z_2 symmetries on $M^4 \times M^1$ and four Z_2 symmetries on $M^4 \times M^1 \times M^1$. As representatives, we discuss the $N = 1$ supersymmetric $SU(5)$ model on the space-time $M^4 \times M^1$, where the standard model fermions can be in the bulk or on the 3-brane, the $N = 2$ supersymmetric $SU(6)$ and $SO(10)$ models on the space-time $M^4 \times S^1 \times S^1$ and the $N = 2$ supersymmetric $SU(6)$ model with gauge–Higgs unification on the space-time $M^4 \times S^1/Z_2 \times S^1/Z_2$, where the standard model fermions must be on the 4-brane, or 3-brane, or 4-brane intersection. For the zero modes, we have 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^{n-3}$ gauge symmetry in which n is the rank of the GUT gauge group. The gauge symmetry and supersymmetry may be broken on the 3-branes, or 4-branes, or 4-brane intersections. In particular, in those models, the extra dimensions can be large and the masses of KK states can be set arbitrarily heavy.

In addition, we discuss the discrete Z_n symmetry on the space-time $M^4 \times A^2$ and $M^4 \times D^2$ in which n is any positive integer. In this kind of scenario, we can break any $SU(M)$ gauge symmetry for $M \geq 5$ down to the $SU(3) \times SU(2) \times U(1)^{M-4}$ gauge symmetry by introducing the global Z_n symmetry as long as n is large enough. In general, considering the 6-dimensional $N = 2$ supersymmetry, we have 4-dimensional $N = 1$ supersymmetry and $SU(3) \times SU(2) \times U(1)^{M-4}$ gauge symmetry in the bulk and on the 4-branes for the zero modes, and on the 3-brane at the origin where only the zero modes exist in the disc D^2 scenario. Including all the KK states, we will have 4-dimensional $N = 4$ supersymmetry and $SU(M)$ gauge symmetry in the bulk, and on the 4-branes. The standard model fermions should be on the boundary 4-brane or 3-brane at the origin. By the way, if we put the standard model fermions on the 3-brane at the origin, the extra dimensions can be large and the gauge hierarchy problem can be solved for there does not exist a proton decay problem at all. Moreover, if the extra space manifold is the annulus A^2, for suitable choices of the inner radius and outer radius we might construct the models where only a few KK states are light and the other KK states are relatively heavy due to the boundary conditions on the inner and outer boundaries, so we might produce the light KK states of gauge fields at future colliders, which is very interesting in collider physics.

If the extra space manifold is a sector of D^2 or a segment of A^2, we point out that the masses of KK states can be set arbitrarily heavy if the range of the angle is small enough.

Furthermore, we discuss the complete global discrete symmetry on the space-time $M^4 \times T^2$. We prove that the possible global discrete symmetries on the torus is Z_2, Z_3, Z_4, and Z_6. We also discuss the 6-dimensional $N = 2$ supersymmetric $SU(5)$ models on the space-time $M^4 \times T^2$ with Z_6 symmetry. There is 4-dimensional $N = 1$ supersymmetry and the standard model gauge symmetry in the bulk for the zero modes, and on the 3-brane at the Z_6 fixed point for all the modes. Including the KK states, we will have the 4-dimensional $N = 4$ supersymmetry and $SU(5)$ gauge symmetry in the bulk, the 4-dimensional $N = 1$ supersymmetry and $SU(5)$ gauge symmetry on the 3-branes at the Z_3 fixed points, and the 4-dimensional $N = 4$ supersymmetry and $SU(3) \times SU(2) \times U(1)$ gauge symmetry on the 3-branes at the Z_2 fixed points. The standard model fermions and Higgs fields can be on any 3-brane at one of the fixed points. In particular, if we put the standard model fermions and Higgs fields on the 3-brane at the Z_6 fixed point, the extra dimensions can be large and the gauge hierarchy problem can be solved because there is no proton decay problem at all.

The phenomenology in those scenarios deserves further study.

Acknowledgements. This research was supported in part by the U.S. Department of Energy under Grant No. DOE-EY-76-02-3071.

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